Speculation and Leverage

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Abstract

Speculative episodes typically involve leverage. For example, the well known tulip-mania episode was accompanied by the introduction of forward contracts which allowed speculators to take leveraged positions in tulip bulbs. More recently, the Great Crash of 1929 was exacerbated by leveraged trusts which used leverage to buy stocks. These leveraged trusts could in turn be bought on margin which allowed speculators to hold highly leveraged positions. More recently, speculation in the housing market was accompanied by extreme leverage. In this paper, I provide an extension of the Harrison-Kreps(1978) speculative model but require the market for borrowing to clear. Clearing the market for borrowing provides endogenous margin requirements which limit borrowing. In addition, the wealth distribution of speculators now becomes a determinant of speculative premia and a fairly rich set of speculative dynamics arise.

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1 Introduction

Recent events in financial markets suggest the importance of understanding how speculation and leverage are linked. However, the current literature on speculation typically considers partial equilibrium models in which the market for borrowing and lending does not clear – in other words the there is more lending than borrowing or vice-versa. This approach is justified by either assuming infinitely wealthy speculators or unlimited borrowing at a fixed risk free rate. Either assumption implies speculators do not face capital limits nor do they face budget constraints. These types of assumptions may be reasonable when studying a small market but become less reasonable in trying to understand how recent events are linked to speculation and leverage.

Several papers have investigated speculation in financial markets. Miller(1978) presents a simple static model to argue that if there are impediments to short selling and investors have heterogeneous views on asset values, then only those who have the highest valuations will end up buying assets. As a result, assets will tend to be valued higher than the average valuation. While this is an important insight, the static nature of the argument and the partial equilibrium nature of the argument does not yield much insight into how, for example, interest rates might interact with influence leverage through time.

Harrison and Kreps(1978) present a model of speculative investor behavior. In a model where again there are impediments to short selling, they show that even if an investor has a lower view of an asset’s value, the right to resell the asset at a future date might provide a significant source of value to the investor and induce them to buy the asset with the intention of reselling it in the future. In this sense, they provide a formal model of the Keynesian notion of speculation. However, in their analysis they assume investors are infinitely wealthy which seems inconsistent with the basic economic principle of scarcity of resources.

The spirit of our analysis is similar to that in Loewenstein and Willard(2006) in which the ability of noise traders to create violations of the law of one price is shown to be inconsistent with market clearing and finite resources. Although this setting is quite different, many of the results stem from market clearing. In particular, a speculative episode is limited by the resources available to the speculators.

In this paper I first sketch a simple extension of the Harrison and Kreps(1978) speculative model. While this extension is of independent interest, the primary motivation for this extension is to develop a suitable benchmark to examine how clearing the borrowing and lending market impacts speculative prices.

The basic uncertainty is generated by a simple (perhaps the simplest) continuous time markov chain. For simplicity there are two states of nature possible at each time – low dividends or high dividends. This leads to jumps in asset prices. While many papers have examined heterogeneous beliefs in a continuous information setting, this paper examines an equilibrium when disagreements are generated by jump uncertainty. Investors must post enough collateral to ensure that they do not jump

1Essentially, this is a continuous time limit of the model considered in Morris(1996)
to default. This stands in contrast to most continuous time equilibrium models with heterogeneous beliefs in the literature. Those models typically have continuous prices which effectively means that the equilibrium can be enforced with zero margin requirements.

At each point in time, risk neutral speculators disagree on the length of time the economy will stay in the current state. Speculators disagreements are generated by heterogeneous priors. Our leading case is where investors do not learn through time as in the original Harrison Kreps model. As an extension we also consider the case where time progresses, investors learn from the past and update their priors according to Bayes rule. The specification of the prior beliefs allows for investors to put more weight on the data or less weight on the data so the speed of learning can be investigated.

When investors are infinitely wealthy or can borrow arbitrary amounts but face short sales constraints as in the original Harrison Kreps model, then fixing the mean belief, prices are increasing in the magnitude of the disagreement; this is expected based on the original Harrison Kreps analysis. However, the percentage price changes are decreasing as disagreement gets large. This is contrary to the assertion that volatility is raised by dispersion of beliefs. In particular, price volatility is highest when investors agree. To understand this effect, recall that in the Harrison Kreps model prices reflect the private valuation plus the option to resell the asset. When the state shifts from high to low dividend, price changes reflect the impact of the change in private valuation to the investor selling the asset but the sale price is higher due to the speculative premium. This then lowers the percentage price change.

The impact of limited capital provides several interesting effects not found in speculative models where speculators have unlimited capital. Prices show pronounced trends punctuated by unpredictable jumps. Prices tend to be highest when speculators are all well capitalized. In this case investors have plenty of collateral to put on large speculative positions. However, as uncertainty unfolds, events tend to favor certain investors who accumulate more wealth. In contrast to the Harrison Kreps model, the wealthy investors become the marginal investor who sets prices. However, the less wealthy investors do not go bankrupt in our model. This is because prices set by the wealthy investors reflect very profitable opportunities for speculation. The less wealthy investors anticipate these prices and optimally reserve some wealth for this possibility.

However, leverage does not perfectly track price levels. In particular, in the good state leverage is highest well after the peak prices occur. This is because volatility falls as prices fall beyond the peak. Because volatility gets lower, speculators can take highly levered positions without defaulting. When disagreements are vary large, volatility can hit zero and we see phenomena similar to Hart’s original example showing how then asset span can collapse.

There are dramatic effects in the term structure of interest rates. Just after the peak stock price is attained, the term structure of interest rates is steeply upward sloping.

A recent literature examines speculation and default and links this activity to asset prices. Our model can help clarify and refine this analysis. First, speculative
prices do not have any direct link to default. In fact it is the lack of default which supports speculative prices. However, speculators do not default in our equilibrium. Second, our model indicates there could be long periods in which there is seemingly no speculative activity. During this time one group builds capital which allows progressively larger positions. At a certain point prices rise dramatically. In effect, when speculators do not eventually agree the economy will cycle through periods of little speculative activity followed by a dramatic price rise.

Finally we examine long run survival of agents. In contrast to Blume and Easley(2004) where all Bayesian learners survive in the limit, I show that in the case of risk neutral speculators, survival depends on the ratio of the prior densities evaluated at the truth.

2 A Simple Continous Time Limit of the Harrison Kreps Model

Here is a sketch of a model which is a continuous time limit of the discrete time analysis in Harrison and Kreps(1978). The continuous time limit also offers new insight into the speculative model of Harrison and Kreps(1978).

I consider a market for a dividend paying asset (a share of stock) and riskless borrowing and lending. Trade occurs continuously over an infinite horizon. The dividend paying asset pays an instantaneous dividend at each point in time of \( \delta_t dt \) so the cumulative dividends paid over an interval \([0, t]\) are given by \( \int_0^t \delta_s ds \). Investors can also borrow and lend but face short sales constraints on the dividend paying asset. The evolution of the dividend is described by a simple regime shifting model. For simplicity assume there are two possible states of the world (0 and 1) at each time. In state 1, \( \delta(1) = 1 \) and in state 0, \( \delta(0) = 0 \). The regime shifts whenever a counting process \( N_t \) jumps.

We suppose there are two investor classes, A and B, who differ in their assessments of how long the regime is likely to last. Our leading case is where the investors have dogmatic beliefs. Specifically, for \( i = A, B \)

\[
P^i[N_1 = n] = e^{-\lambda_i t} \frac{(\lambda_i t)^n}{n!} \tag{2.1}
\]

This specification assumes investors do not update their beliefs no matter how much data they observe. Our analysis is easily be extended to the case where the investors learn from the data. In this case the intensity of the counting process is a function \( \lambda_i(N, t) \). Section 4 examines this case in detail.

Each investor class \( i = A, B \) would like to maximize the expected value of discounted payoffs from trade

\[
E^i \left[ \int_0^\infty e^{-\gamma t} c_t dt \right]
\]

where \( c_t \) represents the nonnegative payoffs from a particular trading strategy.

Given this setup we can compute the private valuation of the stock dividends for an investor under the assumption there is no trade. We denote this private valuation
by $S^A(0, n, t)$ for investor class A in state 0, $S^A(1, n, t)$ for investor class A in state 1, $S^B(0, n, t)$ for investor class B in state 0, and $S^B(1, n, t)$ for investor class B in state 1. When investors do not learn from the data we have

$$S^i(1, n, t) = \int_0^\infty e^{-\gamma t} P^i \{N_t - n \text{ even}\} dt = \frac{\lambda_i + \gamma}{\gamma(\gamma + 2\lambda_i)}$$ (2.2)

$$S^i(0, n, t) = \int_0^\infty e^{-\gamma t} P^i \{N_t - n \text{ odd}\} dt = \frac{\lambda_i}{\gamma(\gamma + 2\lambda_i)}$$ (2.3)

### 2.1 Trade

When we allow trade in the risky asset, a speculative premium arises in the sense that agents will pay more than their private valuation because they anticipate selling the asset in the future for an inflated value. Here we show the equilibrium in the Harrison Kreps setting, assuming infinitely wealthy investors, or unlimited borrowing and lending at the continuously compounded rate $\gamma$. Define $R_t = e^{\gamma t}$.

**Choice Problem 2.1 (Choice Problem In Harrison Kreps).** Given securities endowments $\theta^i$, choose adapted securities holdings $\theta_t$ and $\alpha_t$ to maximize

$$E^i \left[ \int_0^\infty e^{-\gamma t} c_t^i dt \right]$$

subject to

$$dW_t^i = \alpha_t - dR_t + \theta_{t-} dS_t + \theta_{t-} \delta_t dt - c_t^i dt$$

$$\alpha_t \geq 0, \quad \theta_{t-} \geq \theta,$$

and

$$\lim_{t \to \infty} E^i \left[ e^{-\gamma t} W_t^i \right] = 0$$

where $W_t^i = \alpha_t R_t + \theta_t S_t$ and $W_0 = \theta_0 S_0$.

**Proposition 2.1.** Suppose the stock price is bounded. A necessary condition for an optimal solution is the process $e^{-\gamma t} S_t + \int_0^t e^{-\gamma s} \delta_s ds$ is a $P^i$ supermartingale.

**Proof.** Suppose there exist bounded stopping times $\sigma$ and $\tau$ with $\sigma < \tau$ and

$$E^i \left[ e^{-\gamma \tau} S_\tau + \int_0^\tau e^{-\gamma s} \delta_s ds \bigg| \mathcal{F}_\sigma \right] > e^{-\gamma \sigma} S_\sigma + \int_0^\sigma e^{-\gamma s} \delta_s ds$$ (2.4)

or equivalently

$$E^i \left[ e^{-\gamma (\tau - \sigma)} S_\tau + \int_\sigma^\tau e^{-\gamma (s - \sigma)} \delta_s ds \bigg| \mathcal{F}_\sigma \right] > S_\sigma$$ (2.5)

which implies

$$E^i \left[ e^{-\gamma (\tau - \sigma)} (S_\tau - e^{\gamma (\tau - \sigma)} S_\sigma) + \int_\sigma^\tau e^{-\gamma (s - \sigma)} \delta_s ds \bigg| \mathcal{F}_\sigma \right] > 0$$ (2.6)
The trade borrow $S_\sigma$ and buy one share of stock at time $\sigma$; liquidate the position at time $\tau$, deposit the proceeds in the riskless asset and consume $\gamma(S_\tau - e^{\gamma(\tau-\sigma)}S_\sigma)$ forever is feasible as an incremental trade from any candidate optimum. Notice from time $\tau$ on wealth is constant and equal to $S_\tau - e^{\gamma(\tau-\sigma)}S_\sigma$ which satisfies the transversality constraint. Since this trade can be undertaken at any fixed scale there cannot be an optimal solution.

2.1.1 Harrison Kreps Equilibrium

We begin with a fairly general specification of stock price

$$dS_t = (\gamma S_t - \delta_t - \lambda_t^Q \Delta S_t)dt + \Delta S_t dN_t$$

where $\Delta S_t \equiv S_t - S_{t-}$. The discounted stock price plus discounted cumulative dividend

$$de^{-\gamma t}S_t + e^{-\gamma t}\delta_t dt = e^{-\gamma t}S_t \left( dN_t - \lambda_t^Q dt \right)$$

is a $Q$ martingale under a probability measure for which the counting process $N_t$ has intensity $\lambda_t^Q$. This can be rewritten

$$de^{-\gamma t}S_t + e^{-\gamma t}\delta_t dt = e^{-\gamma t}(\lambda_t(N_t, t) - \lambda_t^Q)\Delta S_t dt + e^{-\gamma t}\Delta S_t (dN_t - \lambda_t(N_t, t)dt)$$

Recall $N_t - \int_0^t \lambda_t(s) ds$ is a $P^i$ martingale. The requirement that this is a $P^i$ supermartingale for each $i = A, B$ amounts to

$$(\lambda_t(N_t, t) - \lambda_t^Q)\Delta S_t \leq 0 \quad i = A, B$$

Market clearing in the stock means

$$(\lambda_t(N_t, t) - \lambda_t^Q)\Delta S_t = 0$$

for some $i$. Therefore to clear the stock market we must have

$$\lambda_t^Q \equiv \lambda^Q(N_t, t) = \max_{\{i\}}[\lambda_i(N_t, t)] \quad \text{when} \quad \Delta S_t > 0$$

and

$$\lambda_t^Q \equiv \lambda^Q(N_t, t) = \min_{\{i\}}[\lambda_i(N_t, t)] \quad \text{when} \quad \Delta S_t < 0$$

This intuitively says in state 0, the investor who has the quickest estimate of when the stock will revert to paying a dividend will value the stock highest and in state 1, the investor who has the slowest estimate of when the state will change will value the stock the highest.

It follows that

$$\frac{\partial S_t}{\partial t} = (\gamma S_{t-} - \delta_t - \lambda_t^Q \Delta S_t) e^{-\gamma t} = ((\gamma + \lambda_t^Q)S_{t-} - \delta_t - \lambda_t^Q S_t)$$

and

$$S(1, t, n) = \int_0^\infty e^{-\gamma s} \lambda^Q(n, s+t) \exp \left(-\int_0^s \lambda^Q(n, u + t) du \right) \left( \frac{1}{\lambda^Q(n, s+t)} + S(0, s + t, n + 1) \right) ds$$
\[ S(0, t, n) = \int_0^\infty e^{-\gamma s} \lambda^Q(n, s + t) \exp \left( - \int_0^s \lambda^Q(n, u + t) du \right) S(1, s + t, n + 1) ds \]  

(2.16)

In general there can be many solutions. For example, if for some function \( M \) such that \( \frac{\partial M(n, t)}{\partial t} = -\lambda^Q(n, t)(M(n+1, t) - M(n, t)) \) then adding \( e^{-\gamma t} M(n, t) \) to any solution will produce a new solution. These solutions can thus have rational asset pricing bubbles which are an artifact of infinitely wealthy investors and the fact that we do not clear markets. See Loewenstein and Willard(2006) for a related discussion. However, in absence of these types of bubbles we should have \( 0 \leq S(\cdot, n, t) \leq \frac{1}{\gamma} \). Therefore it seems natural to focus on bounded solutions. In this case there is a unique solution to these equations.

In general these equations must be solved recursively. In the special case of no learning we have explicit equations.

**Proposition 2.2.** Suppose \( \lambda_A > \lambda_B \). The equilibrium stock price in the Harrison Kreps model is given by

\[
S(0, n, t) = \frac{\lambda_A}{\gamma(\gamma + \lambda_A + \lambda_B)}
\]

(2.17)

\[
S(1, n, t) = \frac{\lambda_A + \gamma}{\gamma(\gamma + \lambda_A + \lambda_B)}
\]

(2.18)

and the equilibrium stock price is independent of \( N \) and \( t \).

Stock price percentage jumps are given by

\[
\frac{\Delta S(0, N, t)}{S(0, N, t)} = \frac{\gamma}{\lambda_A}
\]

(2.19)

\[
\frac{\Delta S(1, N, t)}{S(1, N, t)} = -\frac{\gamma}{\lambda_A + \gamma}
\]

(2.20)

Stock price volatility is related to the jump magnitudes and of course the “true” value of the counting process intensity. The jump magnitudes are determined by the investor with the highest \( \lambda_i \). As a result, fixing the mean belief, prices go up and volatility decreases as beliefs diverge. Notice that this also implies the volatility of prices is lower when the speculative premium is highest. The intuition for this result is that as beliefs diverge, a greater portion of the value is due to the resale of the stock and not fundamentals. This then buffers the stock price from changes in the fundamentals. Also it is worth noting that these results do not depend on how many speculators there are and their beliefs; everything is determined by the highest and lowest \( \lambda \).

Table 1 displays the values for equilibrium prices in various settings.

### 3 Speculation with Limited Capital

A funny feature of the above model is that speculators can lose a lot. The optimal solution in equilibrium will involve negative wealth, in which case speculators may
Table 1: Comparison of Valuation.
This table shows the private valuations for Investor Class A and B and the Harrison Kreps equilibrium prices. All cases use $\gamma = 0.1$.

<table>
<thead>
<tr>
<th>Case</th>
<th>S(1,0,0)</th>
<th>S(0,0,0)</th>
<th>$\Delta S/S(1)$</th>
<th>$\Delta S/S(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_A = 0.6$</td>
<td>5.38</td>
<td>4.62</td>
<td>-0.14</td>
<td>0.17</td>
</tr>
<tr>
<td>$\lambda_B = 0.5$</td>
<td>5.45</td>
<td>4.55</td>
<td>-0.17</td>
<td>0.20</td>
</tr>
<tr>
<td>HK</td>
<td>5.83</td>
<td>5</td>
<td>-0.14</td>
<td>0.17</td>
</tr>
<tr>
<td>$\lambda_A = 1$</td>
<td>5.23</td>
<td>4.76</td>
<td>-0.09</td>
<td>0.10</td>
</tr>
<tr>
<td>HK</td>
<td>6.85</td>
<td>6.25</td>
<td>-0.09</td>
<td>0.10</td>
</tr>
</tbody>
</table>

choose to optimally default. Moreover, the equilibrium prices above are the same regardless of whether there are no short sales involved or if a large but finite amount of short sales are permitted. In this section we examine an alternative model in which markets clear and speculators are optimally not defaulting. We begin with the choice problem. We do not prohibit short sales explicitly. However, we explicitly constrain wealth to be nonnegative which will implicitly limit short sales. We show in a later section even if investors are allowed to have some negative wealth (and hence be able to walk away from their debts), they will optimally choose not to do so. The restriction to nonnegative wealth can be thought of as a margin constraint which perfectly anticipates price changes.

Once again we postulate candidate equilibrium prices only now we allow for a possibly stochastic locally riskless interest rate $r_t$. We suppose the stock price is given by

$$dS_t = (r_t S_t - \delta_t - \lambda Q_t \Delta S_t)dt + \Delta S_t dN_t$$

The equity premium for individual $i$ is given by $\frac{(\lambda_i - \lambda_i^0)\Delta S_t}{S_t}$ which can be positive, negative or 0.

Let $R_t = e^{\int_0^t r_s ds}$.

**Choice Problem 3.1** (Choice Problem with Limited Resources). *Given securities endowments $\theta^i$, choose adapted securities holdings $\theta_t$ and $\alpha_t$ to maximize*

$$E^i \left[ \int_0^\infty e^{-\gamma t} c^i_t dt \right]$$

subject to

$$dW^i_t = \alpha_t dR_t + \theta_t dS_t + \theta_t \delta_t dt - c^i_t dt$$

and

$$W_t \geq 0 \Rightarrow \theta_t \Delta S_t \geq -W_t$$

where $W^i_t = \alpha_t R_t + \theta_t S_t$ and $W_0 = \theta^i S_0$.

**Proposition 3.1.** The value function is given by

$$V^i(W, N, t) = W h^i(N, t)$$

$$V^i(W, N, t) = W h^i(N, t)$$
If an optimal solution does not involve default, the function \( h^i(N, t) \geq 1 \) satisfies

\[
\frac{\partial h^i(N, t)}{\partial t} + h^i(N, t)(r_i - \gamma - \lambda_i(N, t)) + \lambda_i(N, t)h^i(N + 1, t) = 0
\]

(3.23)

\[
\lambda_i(N, t)h^i(N + 1, t) = \lambda^Q(N, t)h^i(N, t)
\]

(3.24)

The optimal consumption satisfies

\[
c_t(1 - h^i(N, t)) = 0
\]

(3.25)

**Corollary 3.1.** If the optimal solution does not involve default we have \( e^{-\gamma t}h^i(N_t, t)S_t + \int_0^t e^{-\gamma s}h^i(N_s, s)\delta_{\nu}ds \) and \( e^{\int_0^t(r_i - \gamma)ds}h^i(N_t, t) \) are \( \mathbb{P}^i \) martingales. Therefore if \( \Delta S_t \neq 0 \) \( \lambda^Q \) is unique and we have

\[
\frac{e^{\int_0^t(r_i - \gamma)ds}h^i(N_t, t)}{h^i(0, t)} = \frac{dQ}{dP}\bigg|_t \equiv M_i^t
\]

(3.26)

Moreover, if \( r_i \leq \gamma \), \( h^i(N_t, t) \) is a \( \mathbb{P}^i \) submartingale and if \( r_i = \gamma \), \( h^i(N_t, t) \) is a \( \mathbb{P}^i \) martingale. Therefore, if \( r_i = \gamma \), if \( h^i(N_t, \tau) = 1 \) then \( h^i(N_t, t) = 1 \) for all \( t \geq \tau \). Moreover, we have

\[
W^i(0) = \frac{1}{h^i(0, t)}E^i \left[ \int_0^\infty e^{-\gamma t}c_t1_{\{h^i(N_t, t) = 1\}}dt \right]
= E^Q \left[ \int_0^\infty e^{-\int_0^t r_i ds}c_t1_{\{h^i(N_t, t) = 1\}}dt \right]
= E^i \left[ \int_0^\infty e^{-\int_0^t r_i ds}M_t^i c_t1_{\{h^i(N_t, t) = 1\}}dt \right]
\]

(3.27)

The function \( h^i \) impounds the future speculative opportunities. It represents the value of one unit of wealth optimally invested, or the marginal utility of wealth. In an i.i.d. framework, \( h^i = 1 \) or there is no optimal solution. However, it is important to emphasize that when \( r_t \leq \gamma \), then \( h^i(N_t, t) \) is a submartingale. In effect, the submartingale property lowers the discount rate on future wealth to reflect future speculative opportunities. This is an important observation in our next section.

**Remark 3.1.** The optimal policy involves local indeterminancy in the portfolio and consumption choice. It is important to check the transversality condition

\[
\lim_{t \to \infty} E^i[e^{-\gamma t}h^i(N_t, t)W_t] = 0
\]

(3.28)

which follows from (3.27).

The next result concerns the existence of an optimal solution when returns are i.i.d., or even when returns don’t vary with time.

**Proposition 3.2.** Suppose \( r_t \equiv \gamma \) and \( \lambda^Q \) is constant for \( t \). Then if \( \lambda^Q = \lambda_i \), there exists an optimal solution to the choice problem 3.1 and \( h^i(N, t) = 1 \). If \( \lambda^Q \neq \lambda_i \), then there is no optimal solution and the value function is infinite.

This result indicates that in equilibrium, returns cannot be constant in each state if agents disagree about the intensity of the Poisson process.
3.1 Equilibrium

Our definition of equilibrium is standard.

Definition 3.1. An equilibrium is an interest rate and stock price such that agents solve choice problem 3.1 and the consumption market, the market for loans, and the stock market all clear $\ell \otimes P^i$ almost surely.

Many models of speculation assume either infinitely wealthy investors or unlimited borrowing at a fixed riskless rate. In addition, these models either use exponential utility or risk neutral preferences. These assumptions rule out wealth effects and this implies any individual can be the marginal buyer of the stock. However, if speculators face lower bounds on wealth, even these preferences will give rise to wealth effects. The next section examines equilibrium restrictions which are only implied by market clearing.

3.2 Leverage and Collateral

In this section we explore equilibrium limits and speculation and leverage given market clearing.

Proposition 3.3. In any equilibrium where markets clear, $0 \leq W^A \leq S^A$ and $0 \leq W^B \leq S^B$. As a result,

$$W^i(\iota, N, t) + \theta^i_t \Delta S(\iota, N, t) \geq S(1-\iota, N+1, t) - S^j(1-\iota, N+1, t)$$

(3.29)

In particular, if $W^A(\iota, N, t) > S^A(1-\iota, N+1, t)$ then $A$ is the marginal buyer of the stock.

The importance of this result is that it relies only on market clearing. Equation (3.29) indicates that if there is a speculative premium when the regime shifts, then the wealth constraint for agent $i$ cannot bind for any agent. In fact, margin requirements can be tighter than simply $\theta \Delta S > -W_i$ whenever a speculative premium persists. This proposition also indicates that when agent $B$ is not wealthy, then investor $A$ must be the marginal buyer of the stock in state 1. This contrasts with the Harrison Kreps equilibrium where the marginal buyer is always the buyer with the highest valuation.

3.2.1 Default

Although our investors have finite marginal utility for zero consumption, they will not default in equilibrium. This they is because if they default, then the remaining investors set the prices. Given these prices, an investor with different beliefs can enjoy very large utility. Therefore rather than default, an investor will optimally set a tiny amount of capital to speculate at very favorable prices. But this then says default cannot be part of the equilibrium.

Proposition 3.4. In equilibrium, the wealth constraint does not bind for either investor.
3.2.2 Equilibrium Prices

Our first result derives the state price density and interest rate for the equilibrium.

**Proposition 3.5.** Suppose $\lambda_A > \lambda_B$. The state price density for investor A is given by

$$\rho^A_t = e^{-\gamma \max(\eta, (1-\eta)Z_t)} \max(\eta, 1-\eta) \tag{3.30}$$

where

$$Z_t = e^{(\lambda_A - \lambda_B) t} \left( \frac{\lambda_B}{\lambda_A} \right)^{N_t} \tag{3.31}$$

and $\eta$ is the solution to

$$\theta^A E^A \left[ \int_0^\infty \rho^A_t \delta_t dt \right] = E^A \left[ \int_0^\infty \rho^A_t \delta_t 1_{\{\eta > (1-\eta)Z_t\}} dt \right]. \tag{3.32}$$

The equilibrium interest rate is given by

$$r_t = \begin{cases} \gamma & \text{If } \eta > (1-\eta)Z_t \leq \gamma + \lambda_B - \frac{\eta \lambda_A}{(1-\eta)Z_t} \text{ If } (1-\eta)Z_t > \gamma \text{ and } \frac{\lambda_B}{\lambda_A} (1-\eta)Z_t < \eta \\ \gamma & \text{If } (1-\eta)Z_t > \eta \text{ and } \frac{\lambda_B}{\lambda_A} (1-\eta)Z_t > \eta \end{cases} \tag{3.33}$$

The equilibrium $\lambda^Q$ is given by

$$\lambda^Q_t = \begin{cases} \lambda^A & \text{If } \eta > (1-\eta)Z_t \leq \frac{\eta \lambda_A}{(1-\eta)Z_t} \text{ If } (1-\eta)Z_t > \eta \text{ and } \frac{\lambda_B}{\lambda_A} (1-\eta)Z_t \leq \eta \\ \lambda_B & \text{If } (1-\eta)Z_t > \eta \text{ and } \frac{\lambda_B}{\lambda_A} (1-\eta)Z_t > \eta \end{cases} \tag{3.34}$$

In contrast to the Harrison Kreps model where stock prices can be thought of an investor’s private valuation plus the option to resell the asset, in this setting consumption is priced as the private valuation of the consumption plus the option to resell consumption. Notice the riskfree rate satisfies the bounds $\gamma + \lambda_B - \lambda_A \leq r_t \leq \gamma$. The price of risk satisfies $\lambda_B \leq \lambda^Q \leq \lambda_A$.

There are three important regions for understanding equilibrium pricing. When $\eta > (1-\eta)Z_t$, $\lambda^Q_t = \lambda^A$ and $r_t = \gamma$. In this region, there is no equity premium for investor class A, but investor class B has a risk premium of $(\lambda_B - \lambda_A) \Delta S$. This is positive when $\Delta S < 0$ and negative when $\Delta S > 0$. In this region investor class A consumes the dividend, while investor class B saves and builds capital.

The next region is when $(1-\eta)Z_t > \eta > (1-\eta)Z_t \frac{\lambda_B}{\lambda_A}$. In this region, $r_t < \gamma$ and both investors have a non-zero equity premium. Investor class B consumes the dividend and investor class A saves.

The final region is when $(1-\eta)Z_t \frac{\lambda_B}{\lambda_A}$. In this region, $r_t = \gamma$ and investor A has a non-zero equity premium while investor B does not. In this region investor class B consumes the dividend and investor class A saves.

Intuitively, these regions correspond when A dominates the wealth distribution, a transition region, and when B dominates the wealth distribution. If the economy
begins in the first region, then as time passes this tends to favor investor class B. If the state doesn’t change then the economy will transition through the three regions. However, changes in state lower $(1 - \eta)Z_t$ so state transitions will tend to bring the economy back to region 1. State transitions favor investor class A so state transitions tend to result in A dominating the wealth distribution.

The next proposition summarizes asset prices when all markets clear.

**Proposition 3.6.** Suppose $\lambda_A > \lambda_B$. Equilibrium prices are given by

$$S(1, N, t) = \frac{\eta}{\max(\eta, (1 - \eta)Z_t)} \sum_{j=0}^{\infty} \int_{0}^{t^*(2j,Z_t)} e^{-(\gamma + \lambda_A)s} \frac{(\lambda_A s)^{2j}}{(2j)!} ds$$

$$+ \frac{(1 - \eta)Z_t}{\max(\eta, (1 - \eta)Z_t)} \sum_{j=0}^{\infty} \int_{t^*(2j,Z_t)}^{\infty} e^{-(\gamma + \lambda_B)s} \frac{(\lambda_B s)^{2j}}{(2j)!} ds$$

where

$$t^*(j, Z_t) = \frac{j \ln \left( \frac{\lambda_B}{\lambda_A} \right) - \ln \left( \frac{\eta}{(1 - \eta)Z} \right)}{\lambda_B - \lambda_A} \vee 0$$

The next proposition gives the equilibrium wealth processes.

**Proposition 3.7.** The equilibrium wealth process for investor class B is given by

$$W^B(1, N, t) = \frac{(1 - \eta)Z_t}{\max(\eta, (1 - \eta)Z_t)} \sum_{j=0}^{\infty} \int_{0}^{t^*(2j+1,Z_t)} e^{-(\gamma + \lambda_B)s} \frac{(\lambda_B s)^{2j+1}}{(2j+1)!} ds$$

$$W^B(0, N, t) = \frac{(1 - \eta)Z_t}{\max(\eta, (1 - \eta)Z_t)} \sum_{j=0}^{\infty} \int_{t^*(2j+1,Z_t)}^{\infty} e^{-(\gamma + \lambda_B)s} \frac{(\lambda_B s)^{2j+1}}{(2j+1)!} ds$$

The equilibrium wealth process for investor class A can be obtained from market clearing:

$$W^A(\cdot, N, t) = S(\cdot, N, t) - W^B(\cdot, N, t)$$

**Corollary 3.2.**

$$\frac{\partial S(\cdot, N, t)}{\partial \eta} = \begin{cases} -\frac{1}{\eta(1 - \eta)Z} \frac{W^B(\cdot, N, t)}{\lambda_B - \lambda_A} & \text{if } \eta > (1 - \eta)Z \\ \frac{1}{\eta(1 - \eta)Z} \frac{W^A(\cdot, N, t)}{\lambda_B - \lambda_A} & \text{if } (1 - \eta)Z > \eta \end{cases}$$
The stock price is highest when $\eta = (1 - \eta)Z_t$. In addition, when $\eta \uparrow 1$ the stock price goes to A’s private valuation and as $\eta \downarrow 0$ the stock price goes to B’s private valuation.

\[
\frac{\partial S(\cdot, N, t)}{\partial t} = (\lambda_A - \lambda_B)W^B(\cdot, N, t)1_{(1 - \eta)Z_t < \eta} + (\lambda_B - \lambda_A)W^A(\cdot, N, t)1_{(1 - \eta)Z_t > \eta}
\]
\[
= \left( r_t + \lambda_t^Q - \gamma - \lambda_A \frac{W^A}{S} - \lambda_B \frac{W^B}{S} \right) S \quad (3.42)
\]

Corollary 3.2 indicates the stock price will unambiguously increase when $\eta > (1 - \eta)Z_t$ and decrease when $(1 - \eta)Z_t > \eta$. The maximal stock price is when $\eta = (1 - \eta)Z_t$.

Figure 1 shows the equilibrium stock price versus time. The upper line shows how the stock price moves in state 1 while the bottom graph shows the resulting stock price when the state changes from one to zero. Notice the magnitude of the price change is the difference between the two lines. The magnitude of the changes grows until the peak, then shrinks, and then grows again. This will be examined in more detail later, but this behavior is important to understand how leverage and prices are linked. Figure 2 shows leverage in the two regimes. Prior to the peak, as investor class B builds capital he can take more leverage in state 1 despite the increase in volatility. However, in State 1 leverage peaks well after the peak prices. This is because the volatility shrinks after the peak prices; lowered volatility allows bigger leverage.

Figure 3 shows the stock price versus time but for a big disagreement. Here we see the stock price volatility can hit zero and even move so that when the dividend drops the stock price goes up. Figure 4 shows leverage in state 1. When the stock price volatility vanishes, the the speculators take infinite positions. This is similar to the non-existence example in Hart (1975) but here because this only occurs on a zero measure set the stochastic integrals describing trading gains are defined. It is a bit of an abstraction to say equilibrium exists here, however.

Figures 5, 6, 7, and 8 show how asset prices in both states at time 0 behave as a function of $\eta$ for small disagreement and larger disagreement. When $\eta$ is large, Investor class A dominates the wealth distribution and prices reflect A’s private valuation. When $\eta$ goes to $\frac{1}{2}$ prices go up to the maximal value which greatly exceeds each individuals private valuation and when $\eta$ gets small, prices reflect Investor class B’s private valuation. Loosely speaking, as more time is spent in each state, prices will tend to move in the direction of $\eta$ getting smaller since Investor class A always estimates the state shifting faster than Investor class B, the more time spent in a given state tends to shift the wealth distribution in B’s favor.

**Remark 3.2.** We can easily generalize this to the case where the stock pays a dividend $\delta_t(1) = H$ and $\delta_t(0) = L$. The stock price in state 1 will be

\[
HS(1, N, t) + LS(0, N, t)
\]

and in state 0 the stock price will be

\[
LS(1, N, t) + HS(0, N, t)
\]

(3.43)
Figure 1: Price vs. Time
Parameters: $\lambda_A = 0.6$, $\lambda_B = 0.5$, $\gamma = 0.1$.

Figure 2: Leverage vs. Time
Parameters: $\lambda_A = 0.6$, $\lambda_B = 0.5$, $\gamma = 0.1$. 
Figure 3: Price vs. Time
Parameters: $\lambda_A = 1.0$, $\lambda_B = 0.5$, $\gamma = 0.1$.

Figure 4: Leverage vs. Time
Parameters: $\lambda_A = 1.0$, $\lambda_B = 0.5$, $\gamma = 0.1$. 
Figure 5: Stock Price in State 1 as a Function of $\eta$.
This figure shows how the stock price varies with $\eta$ in state 1. Parameters: $\lambda_A = 0.6$, $\lambda_B = 0.5$, $\gamma = 0.1$.

Figure 6: Stock Price in State 0 as a Function of $\eta$.
This figure shows how the stock price varies with $\eta$ in state 0. Parameters: $\lambda_A = 0.6$, $\lambda_B = 0.5$, $\gamma = 0.1$. 
Figure 7: Stock Price in State 1 as a Function of $\eta$.
This figure shows how the stock price varies with $\eta$ in state 1 when agents have larger disagreement. Parameters: $\lambda_A = 1.0$, $\lambda_B = 0.5$, $\gamma = 0.1$.

Figure 8: Stock Price in State 0 as a Function of $\eta$.
This figure shows how the stock price varies with $\eta$ in state 1 when agents have larger disagreement. Parameters: $\lambda_A = 1.0$, $\lambda_B = 0.5$, $\gamma = 0.1$.  

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3.2.3 Equilibrium Stock Price Volatility

In this section, we look at the equilibrium stock price changes.

Proposition 3.8. Equilibrium percentage stock price jumps are given by

\[
\frac{\Delta S(\cdot, N, t)}{S(\cdot, N, t)} \equiv \frac{S(1-\cdot, N+1, t) - S(\cdot, N, t)}{S(\cdot, N, t)} = \left(\frac{\gamma + \lambda_A}{\lambda^Q t} - 1\right) \frac{W^A(\cdot, N, t)}{S(\cdot, N, t)} + \left(\frac{\gamma + \lambda_B}{\lambda^Q t} - 1\right) \frac{W^B(\cdot, N, t)}{S(\cdot, N, t)} - \frac{\delta(\cdot)}{\lambda^Q t S(\cdot, N, t)}
\]

\[
= \frac{\gamma}{\lambda^Q t} - \frac{\delta(\cdot)}{\lambda^Q t S(\cdot, N, t)} + \left(\frac{\lambda_A W^A(\cdot, N, t)}{\lambda^Q t S(\cdot, N, t)} + \frac{\lambda_B W^B(\cdot, N, t)}{\lambda^Q t S(\cdot, N, t)} - 1\right) \quad (3.45)
\]

Proposition 3.8 indicates there are three distinct regions to analyze to understand the equilibrium stock price jumps. When \(\eta > (1-\eta)Z\) percentage stock price changes unambiguously decrease as \(\eta\) decreases. When \((1-\eta)Z\lambda_A > \eta\), percentage stock price changes also unambiguously decrease as \(\eta\) decreases. When \(\eta \downarrow 0\), percentage stock price changes approach those of an economy populated by investors with B’s beliefs and when \(\eta \uparrow 1\) percentage stock price changes approach those of an economy populated by investors with A’s beliefs. In the middle region, where \((1-\eta)Z\lambda_B < \eta < (1-\eta)Z\), the percentage stock price changes increase as \(\eta\) decreases. This is caused by the fact that when the state changes in this region, the marginal buyer for consumption also changes.

Figures 9, 10, 11, and 12 show the stock price changes as a function of \(\eta\) for a moderate disagreement and a larger disagreement.

In the first region when \(\eta > (1-\eta)Z_t\) stock price jumps are unambiguously
Figure 10: Stock Price Percentage Change in State 0 as a Function of $\eta$.
This figure shows how $\frac{\Delta S}{S}$ varies with $\eta$ in State 0. Parameters: $\lambda_A = 0.6$, $\lambda_B = 0.5$, $\gamma = 0.1$.

Figure 11: Stock Price Percentage Change in State 1 as a Function of $\eta$.
This figure shows how $\frac{\Delta S}{S}$ varies with $\eta$ in State 1 when agents have larger disagreement. Parameters: $\lambda_A = 1.0$, $\lambda_B = 0.5$, $\gamma = 0.1$. 

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Figure 12: Stock Price Percentage Change in State 0 as a Function of $\eta$.
This figure shows how $\frac{\Delta S}{S}$ varies with $\eta$ in State 0 when agents have larger disagreement.
Parameters: $\lambda_A = 1.0$, $\lambda_B = 0.5$, $\gamma = 0.1$.

negative in state 1. We have

$$
\Delta S = \left( \frac{\gamma + \lambda_A}{\lambda_A} - 1 \right) W^A + \left( \frac{\gamma + \lambda_B}{\lambda_A} - 1 \right) W^B - \frac{1}{\lambda_A} < \left( \frac{\gamma + \lambda_A}{\lambda_A} - 1 \right) (W^A + W^B) - \frac{1}{\lambda_A} 
= \frac{\gamma S}{\lambda_A} - \frac{1}{\lambda_A} < 0. \quad (3.46)
$$

In addition we can see that the stock price jumps must become more negative as B’s wealth increases, in other words as $\eta \rightarrow (1 - \eta)Z_t$.

However, large disagreements might cause stock price jumps to be positive when $\eta < (1 - \eta)Z_t$. That is the dividend drops, the stock price goes up. This occurs due to an interest rate effect. When $(1 - \eta)Z_t \lambda_B A > \eta$ we have

$$
\Delta S = \left( \frac{\gamma + \lambda_A}{\lambda_B} - 1 \right) W^A + \left( \frac{\gamma + \lambda_B}{\lambda_B} - 1 \right) W^B - \frac{1}{\lambda_B} 
= \frac{\gamma W^B - 1}{\lambda_B} + \frac{\gamma + \lambda_A - \lambda_B}{\lambda_B} W^A \quad (3.47)
$$

So if $W^A$ is large, then the stock price jump can be positive. But as $\eta \rightarrow 0$, the stock price approaches B’s private valuation so for small values of $\eta$, $W^A$ is small and the stock price jump must be negative in state 1. For intermediate values the magnitude of the stock price jump can be positive or negative in state 1 depending on the magnitude of disagreement.

In state 0, stock price jumps are unambiguously positive when $(1 - \eta)Z_t \lambda_B A > \eta$. 

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We have

\[ \Delta S = \left( \frac{\gamma + \lambda_A}{\lambda_B} - 1 \right) W^A + \left( \frac{\gamma + \lambda_B}{\lambda_B} - 1 \right) W^B \]

\[ > \left( \frac{\gamma + \lambda_B}{\lambda_B} - 1 \right) (W^A + W^B) = \frac{\gamma}{\lambda_B} S > 0 \]  

(3.48)

In addition, when A’s wealth increases, in other words as \((1 - \eta)Z_t \frac{\lambda_B}{\lambda_A} \to \eta\), we have stock price jumps become more positive.

But when \((1 - \eta)Z_t \frac{\lambda_B}{\lambda_A} < \eta\), stock price jumps can be positive or negative in state 0. In this case

\[ \Delta S = \left( \frac{\gamma + \lambda_A}{\lambda_A} - 1 \right) W^A + \left( \frac{\gamma + \lambda_B}{\lambda_A} - 1 \right) W^B \]  

(3.49)

In particular, when \(\gamma + \lambda_B < \lambda_A\) and \(W^B \geq W^A\) then the stock price jump can be negative. In other words, the dividend goes up and the stock price jumps down.

Recalling that \(\gamma + \lambda_B - \lambda_A\) is the lower bound on the real interest rate, we see this possibility corresponds to when the real interest rate can be negative. As \(\eta \to 1\), however, stock prices converge to A’s private valuation so for large values of \(\eta\), \(W^B\) is small and the stock price jumps will be negative.

In general, we see that stock price volatility can be quite different depending on the equilibrium wealth distribution.

### 3.2.4 Equilibrium Portfolio Choice

The next proposition gives the equilibrium portfolio choice for investor class B.

**Proposition 3.9.** Investor B’s equilibrium portfolio choice is given by

\[ \theta_B(1, N, t) \Delta S(1, N, t) = \left( \frac{\gamma + \lambda_B}{\lambda_B^Q} - 1 \right) W^B(1, N, t) \]

\[ - \frac{(1 - \eta)Z_t}{\lambda_A \max(\eta, (1 - \eta)Z_t \frac{\lambda_B}{\lambda_A})} \sum_{j=1}^{\infty} e^{-(\gamma + \lambda_B)\tau(2j, Z_t)} \frac{(\lambda_B \tau(2j + 1, Z_t))^2 j}{(2j)!} - \frac{1}{\lambda_B^Q} 1_{(1 - \eta)Z_t > \eta} \]

(3.50)

\[ \theta_B(0, N, t) \Delta S(0, N, t) = \left( \frac{\gamma + \lambda_B}{\lambda_B^Q} - 1 \right) W^B(0, N, t) \]

\[ - \frac{(1 - \eta)Z_t}{\lambda_A \max(\eta, (1 - \eta)Z_t \frac{\lambda_B}{\lambda_A})} \sum_{j=0}^{\infty} e^{-(\gamma + \lambda_B)\tau(2j + 1, Z_t)} \frac{(\lambda_B \tau(2j + 1, Z_t))^2 j + 1}{(2j + 1)!} \]

(3.51)

Recall, we can think of investor B’s equilibrium consumption as a stream of option cash flows which pay $1 in state 1 whenever \((1 - \eta)Z_t > \eta\). In other words, this is a cash flow stream of digital options with strike price \(\eta\). The sum in Equations (3.50)
and (3.51) represent the value of a derivative security which pays $1 whenever \((1 - \eta)Z_t = \eta\) similar to the hedge ratio for a digital option.

The baseline portfolio choice in the Harrison Kreps equilibrium is fairly static: Investor \(i\) holds the asset whenever \(\lambda_i(N, t)\Delta S_t > \lambda_j(N, t)\Delta S_t\). When speculators have limited resources however, the wealth distribution introduces changes in the marginal buyer. This in turn affects the hedging demands of the speculators who optimally take into account the investment opportunity set going forward. Figures 13, 14, 15, and 16 show the equilibrium portfolio choice at time 0 as a function of \(\eta\). For large values of \(\eta\), Investor class A dominates the wealth distribution and in both states Investor class A is buying shares. For smaller values of \(\eta\) Investor class B dominates the wealth distribution and in both states is buying shares. As \(\eta\) goes to \(\frac{1}{2}\) the wealth distribution equalizes and we see Investors taking larger speculative positions. We see that Investors follow a momentum strategy in one state and a contrarian strategy in the other state. When investors have large disagreements, we see that positions grow unboundedly around the points where \(\Delta S = 0\).

3.2.5 Term Structure

Recall, equilibrium interest rates are less than or equal to the rate of time preference \(\gamma\) and are lowest when the stock price is just past its peak. The reason for this is that investors account for future speculative investments and anticipate that equilibrium prices will offer even greater speculative prices in the bad states of the world.

**Proposition 3.10.** A Zero coupon bond which pays one unit and matures at time \(T\)
Figure 14: Number of Shares of Stock for Investor Class A in State 0 as a Function of \( \eta \).
This figure shows how the number of shares Investor class A holds in State 0 as a function of \( \eta \). Parameters: \( \lambda_A = 0.6, \lambda_B = 0.5 \), and \( \gamma = 0.1 \)

Figure 15: Number of Shares of Stock for Investor Class A in State 1 as a Function of \( \eta \).
This figure shows how the number of shares Investor class A holds in State 1 as a function of \( \eta \) for larger disagreement. Parameters: \( \lambda_A = 1.0, \lambda_B = 0.5, \) and \( \gamma = 0.1 \)
Figure 16: Number of Shares of Stock for Investor Class A in State 0 as a Function of η.
This figure shows how the number of shares Investor class A holds in State 1 as a function of η for larger disagreement. Parameters: λ_A = 1.0, λ_B = 0.5, and γ = 0.1

has price at time t given by

\[
e^{-\gamma(T-t)} \left( \frac{\eta}{\max[\eta, (1-\eta)Z_t]} \sum_{j=n^*}^{\infty} e^{-\lambda_A(T-t)} \frac{\lambda_A^j(T-t)^j}{j!} + \frac{(1-\eta)Z_t}{\max[\eta, (1-\eta)Z_t]} \sum_{j=0}^{n^*-1} e^{-\lambda_B(T-t)} \frac{\lambda_B^j(T-t)^j}{j!} \right)
\]

(3.52)

\[
e^{-\gamma(T-t)} \left( \frac{\eta}{\max[\eta, (1-\eta)Z_t]} \frac{\Gamma(n^*) - \Gamma(n^*, \lambda_A(T-t))}{\Gamma(n^*)} + \frac{(1-\eta)Z_t}{\max[\eta, (1-\eta)Z_t]} \frac{\Gamma(n^*, \lambda_B(T-t))}{\Gamma(n^*)} \right)
\]

(3.53)

and \( n^* \) is given by the smallest nonnegative integer than greater than

\[
\ln \left( \frac{-\eta}{(1-\eta)Z_t} \right) - (\lambda_A - \lambda_B)(T-t)
\]

\[
\ln \left( \frac{\lambda_B}{\lambda_A} \right)
\]

(3.54)

Figures 17 shows the term structure of interest rates when the stock price is highest, that is when \( \eta = \frac{1}{2} \).

4 Learning

In the “true” model the switch from State 0 to State 1 and back is governed by a Poisson process \( N_t \) with constant intensity \( \lambda \). It is standard to assume that \( N_0 = 0 \). At each jump the state changes. Well known results on the Poisson process (see Karlin and Taylor) give

\[
P[N_t = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}
\]

(4.55)

\[
P[N_t \text{ odd}] = \frac{1}{2} - \frac{1}{2} e^{-2\lambda t}
\]

(4.56)
Figure 17: Term Structure.

This figure shows the term structure for parameters $\lambda_A = 0.6$, $\lambda_B = 0.5$, $\gamma = 0.1$, and $\eta = 0.5$.

$$P[N_t \text{ even}] = \frac{1}{2} + \frac{1}{2} e^{-2t}$$ (4.57)

Again for simplicity, assume there are two classes of investors, “A” and “B.” The true value of $\lambda$ is unknown to each investor class. Instead they have prior beliefs on $\lambda$ which are Gamma distributed, that is the prior probability density for $\lambda$ is given by

$$\frac{k_i e^{-k_i \lambda \lambda_i - 1}}{\Gamma(z_i)} \quad i = A, B$$ (4.58)

where $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$. If $k_A \neq k_B$ and/or $z_A \neq z_B$ then agents have heterogeneous priors. Each investor class updates their beliefs given the data using Bayes rule. Thus at time $t$ given $N_t = n$ their posterior belief on $\lambda$ is given by

$$E^i[\lambda|N_t = n] = \frac{\int_0^\infty \lambda e^{-k_i \lambda \lambda t} \lambda e^{-\lambda t \lambda_i n} d\lambda}{\int_0^\infty \lambda e^{-k_i \lambda \lambda t} \lambda e^{-\lambda t \lambda_i m} d\lambda} = \frac{z_i + n}{t + k_i} \equiv \lambda_i(n, t)$$ (4.59)

In particular

$$E^i[\lambda] = \frac{z_i}{k_i}$$ (4.60)

and the process $\lambda_i(N_t, t)$ is a $P^i$ martingale. The process $N_t - \int_0^t \lambda_i(N_s, s) ds$ is a $P^i$ martingale. $\lambda_i(n, t)$ is decreasing in $t$ and increasing in $N$. After an infinite amount of data, $\lambda_i(N_t, t) \to \frac{N_t}{2} \to \lambda$ so in the limit agents learn the truth.

Similar calculations give for $s \leq t$ and $n \geq m$

$$P^i[N_t = n|N_s = m] = \frac{\int_0^\infty \lambda e^{-k_i \lambda \lambda t} \lambda e^{-\lambda t (\lambda_i n - m)} \lambda e^{-\lambda t (\lambda_i m)} d\lambda}{\int_0^\infty \lambda e^{-k_i \lambda \lambda t} \lambda e^{-\lambda t (\lambda_i m)} d\lambda} = \frac{(t-s)^{n-m} \Gamma(n+z_i)(s+k_i)^{z_i+m}}{(n-m)! \Gamma(m+z_i)(t+k_i)^{n+z_i}}.$$ (4.61)
If \( n \) is even we have
\[
P^i[N_t \text{ odd}|N_s = n] = \frac{1}{2} - \frac{1}{2} \frac{(s + k_i)^{m+z_i}}{(2(t-s) + s + k_i)^{z_i+m}}
\]  
(4.62)
\[
P^i[N_t \text{ even}|N_s = n] = \frac{1}{2} + \frac{1}{2} \frac{(s + k_i)^{m+z_i}}{(2(t-s) + s + k_i)^{z_i+m}}
\]  
(4.63)
If \( n \) is odd
\[
P^i[N_t \text{ odd}|N_s = n] = \frac{1}{2} + \frac{1}{2} \frac{(s + k_i)^{m+z_i}}{(2(t-s) + s + k_i)^{z_i+m}}
\]  
(4.64)
\[
P^i[N_t \text{ even}|N_s = n] = \frac{1}{2} - \frac{1}{2} \frac{(s + k_i)^{m+z_i}}{(2(t-s) + s + k_i)^{z_i+m}}
\]  
(4.65)
and
\[
P^i[N_t \text{ odd}] = \frac{1}{2} - \frac{1}{2} \frac{k_i^{z_i}}{(2t + k_i)^{z_i}}
\]  
(4.66)
\[
P^i[N_t \text{ even}] = \frac{1}{2} + \frac{1}{2} \frac{k_i^{z_i}}{(2t + k_i)^{z_i}}
\]  
(4.67)
and in particular
\[
P^i[N_t = n] = \frac{t^n \Gamma(n+z_i)}{n! \Gamma(z_i)} \frac{k_i^{z_i}}{(t+k_i)^{n+z_i}}
\]  
(4.68)

The parameters \( z_i \) and \( k_i \) influence the prior belief on \( \lambda \) and how much weight the investor puts on the data when updating the beliefs. Recall initial expected value of \( \lambda \) is given by \( \lambda_i(0,0) = \frac{\tilde{z}_i}{k_i} \). Fixing the prior belief \( \lambda_i \) and setting \( k_i \equiv z_i \lambda_i \) and letting \( z_i \to \infty \) we obtain the limiting case where the investor puts no weight on the data and does not learn.

On the other hand, letting \( z_i \) and \( k_i \) go to zero, the investor puts more weight on the data and his posterior estimate of \( \lambda \) comes close to the empirical frequency \( \frac{N_t}{t} \). Thus our choice of prior beliefs represents a fairly general class of priors which can accommodate different beliefs as well as differential learning.

Simple computations give the expected present value of dividends for each investor class in each state:
\[
S^i(0,n,t) = E^i \left[ \int_t^\infty e^{-\gamma(s-t)} \delta_s ds | N_t = n \right] = \int_t^\infty e^{-\gamma(s-t)} P^i[N_s = n \text{ odd}|N_t = n] ds
\]
\[
= \int_t^\infty e^{-\gamma(s-t)} \left( \frac{1}{2} - \frac{1}{2} \frac{k_i^{z_i}}{(2s + t + k_i)^{z_i+m}} \right) ds
\]
\[
= \frac{1}{2\gamma} \left( 1 - (t + k_i)^{n+z_i} e^{\gamma(t+k_i)/2} \frac{\Gamma(1-z_i-n, \gamma(t+k_i)/2)}{\Gamma(1-z_i-n)} \right)
\]  
(4.69)
\[
S^i(1,n,t) = E^i \left[ \int_t^\infty e^{-\gamma(s-t)} \delta_s ds | N_t = n \right] = \int_t^\infty e^{-\gamma(s-t)} P^i[N_s = n \text{ even}|N_t = n] ds
\]
\[
= \int_t^\infty e^{-\gamma(s-t)} \left( \frac{1}{2} + \frac{1}{2} \frac{k_i^{z_i}}{(2s + t + k_i)^{z_i+m}} \right) ds
\]
\[
= \frac{1}{2\gamma} \left( 1 + (t + k_i)^{n+z_i} e^{\gamma(t+k_i)/2} \frac{\Gamma(1-z_i-n, \gamma(t+k_i)/2)}{\Gamma(1-z_i-n)} \right)
\]  
(4.70)
where $\Gamma(z,x) = \int_x^\infty e^{-t}t^{z-1}dt$. We note we have the following limits

$$\lim_{t \to \infty} S^i(1, n, t) = \frac{1}{\gamma} \lim_{t \to \infty} S^i(0, n, t) = 0 \quad (4.71)$$

$$\lim_{n \to \infty} S^i(\cdot, n, t) = \frac{1}{2\gamma} \quad (4.72)$$

### 4.0.6 Equilibrium Prices

The next proposition summarizes asset prices when all markets clear.

**Proposition 4.1.** Suppose $\lambda_A(N,t) > \lambda_B(N,t)$ Equilibrium prices are given by

$$S(1, N, t) = \frac{\eta}{\max(\eta, (1-\eta)Z_t)} \sum_{j=0}^\infty t^j(\eta, z_t, N) e^{-\gamma t} \binom{\lambda_j}{\lambda_j(2j)} \left( \frac{\Gamma(N+2j+z_A)}{\Gamma(N+z_A)} \right) \left( \frac{\Gamma(t+k_A)^{N+j}A}{\Gamma(t+k_A)^{N+j}B} \right) ds$$

$$S(0, N, t) = \frac{\eta}{\max(\eta, (1-\eta)Z_t)} \sum_{j=0}^\infty t^j(\eta, z_t, N) e^{-\gamma t} \binom{\lambda_j}{\lambda_j(2j)} \left( \frac{\Gamma(N+2j+z_B)}{\Gamma(N+z_B)} \right) \left( \frac{\Gamma(t+k_B)^{N+j}B}{\Gamma(t+k_B)^{N+j}A} \right) ds$$

where

$$Z_t = \frac{\Gamma(N+z_B)}{\Gamma(N+z_A)} \frac{k_A}{k_B} \left( \frac{\Gamma(t+k_A)^{N+j}A}{\Gamma(t+k_B)^{N+j}B} \right) \quad (4.75)$$

$$dZ_t = \frac{\lambda_B(N,t) - \lambda_A(N,t)}{\lambda_A(N,t)} Z_t - (dN_t - \lambda_A(N,t)dt) \quad Z_0 = 1 \quad (4.76)$$

and $t^j(\eta, z_t, N)$ is defined by the solution to

$$\eta \left( \frac{\Gamma(N+j+z_A)}{\Gamma(N+z_A)} \right) \left( \frac{(t+k_A)^{N+j}A}{t+k_A} \right) = (1-\eta)Z \left( \frac{\Gamma(N+j+z_B)}{\Gamma(N+z_B)} \right) \left( \frac{(t+k_B)^{N+j}B}{t+k_B} \right) \quad (4.77)$$

if a nonnegative solution exists and $t^* = 0$ otherwise. The equilibrium interest rate is given by

$$r_t = \begin{cases} 
\gamma & \text{if } \eta > (1-\eta)Z_t \\
\gamma + \lambda_B - \lambda_A e^{-\gamma \max(\eta, (1-\eta)Z_t)} & \text{if } (1-\eta)Z_t > \eta \text{ and } \frac{\lambda_B}{\lambda_A}(1-\eta)Z_t < \eta \\
\gamma & \text{if } (1-\eta)Z_t > \eta \text{ and } \frac{\lambda_B}{\lambda_A}(1-\eta)Z_t > \eta 
\end{cases} \quad (4.78)$$

**Corollary 4.1.**

$$\frac{\partial S(N,t)}{\partial \eta} = \begin{cases} 
-\frac{1}{(1-\eta)Z} W_B(N,t) & \text{if } \eta > (1-\eta)Z \\
-\frac{1}{\eta(1-\eta)Z} W_A(N,t) & \text{if } (1-\eta)Z > \eta 
\end{cases} \quad (4.79)$$

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Figure 18: Stock Price versus Time.

This figure shows the stock price versus time for \( N = 0 \) and \( N = 1 \). Parameters: \( z_A = z_B = 1 \), \( \lambda_A(0, 0) = 0.6 \), \( \lambda_B(0, 0) = 0.5 \), and \( \gamma = 0.1 \).

The stock price is highest when \( \eta = (1 - \eta)Z \). In addition, when \( \eta \uparrow 1 \) the stock price goes to A’s private valuation and as \( \eta \downarrow 0 \) the stock price goes to B’s private valuation. We also have the following limits

\[
\lim_{t \to \infty} S(1, n, t) = \frac{1}{\gamma} \quad \lim_{t \to \infty} S(0, n, t) = 0 \quad (4.80)
\]

\[
\lim_{n \to \infty} S(\cdot, n, t) = \frac{1}{2\gamma} \quad (4.81)
\]

Figure 18 shows the stock price as a function of time when we start in state 1. The top line is the stock price in state 1 and the bottom line illustrates the stock price when the state shifts to state 0. Stock price jumps are increasing as time passes. Here there are two effects: 1) as time passes, investors tend to update their beliefs and 2) their beliefs tend to agree more. Figure 19 shows leverage versus time in state 1. Not surprisingly, leverage decreases as disagreement decreases.

Figures 20, and 21 show how asset prices in both states at time 0 behave as a function of \( \eta \) for the case of learning. When \( \eta \) is large, Investor class A dominates the wealth distribution and prices reflect A’s private valuation. When \( \eta \) goes to \( \frac{1}{2} \) prices go up to the maximal value which greatly exceeds each individuals private valuation and when \( \eta \) gets small, prices reflect Investor class B’s private valuation. Loosely speaking, as more time is spent in each state, prices will tend to move in the direction of \( \eta \) getting smaller since Investor class A always estimates the state shifting faster than Investor class B, the more time spent in a given state tends to shift the wealth distribution in B’s favor.

4.0.7 Term Structure

Recall, equilibrium interest rates are less than or equal to the rate of time preference \( \gamma \) and are lowest when the stock price is just past its peak. The reason for this is that
Figure 19: Leverage versus time in State 1.
This figure shows leverage versus time for $N = 0$. Parameters: $z_A = z_B = 1$, $\lambda_A(0, 0) = 0.6$, $\lambda_B(0, 0) = 0.5$, and $\gamma = 0.1$.

Figure 20: Stock Price in State 1 versus $\eta$.
This figure shows stock price in state 1 for $N = 0$ versus $\eta$. Parameters: $z_A = z_B = 1$, $\lambda_A(0, 0) = 0.6$, $\lambda_B(0, 0) = 0.5$, and $\gamma = 0.1$. 
Figure 21: Stock Price in State 0 versus $\eta$.
This figure shows stock price in state 0 for $N = 0$ versus $\eta$. Parameters: $z_A = z_B = 1$, $\lambda_A(0,0) = 0.6$, $\lambda_B(0,0) = 0.5$, and $\gamma = 0.1$.

Figure 22: Percentage change in Stock Price in State 1 versus $\eta$.
This figure shows the percentage change in stock price in state 1 for $N = 0$ versus $\eta$. Parameters: $z_A = z_B = 1$, $\lambda_A(0,0) = 0.6$, $\lambda_B(0,0) = 0.5$, and $\gamma = 0.1$. 

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Figure 23: Percentage change in Stock Price in State 0 versus $\eta$.
This figure shows the percentage change in stock price in state 0 for $N = 0$. Parameters: $z_A = z_B = 1$, $\lambda_A(0,0) = 0.6$, $\lambda_B(0,0) = 0.5$, and $\gamma = 0.1$.

Figure 24: Initial Share Holdings for Agent A in State 1.
This figure shows the number of shares of stock for agent A in state 1 for $N = 0$ versus $\eta$. Parameters: $z_A = z_B = 1$, $\lambda_A(0,0) = 0.6$, $\lambda_B(0,0) = 0.5$, and $\gamma = 0.1$. 
Figure 25: Initial Share Holdings for Agent A in State 0.
This figure shows the number of shares of stock for agent A in state 0 for $N = 0$ versus $\eta$. Parameters: $z_A = z_B = 1$, $\lambda_A(0,0) = 0.6$, $\lambda_B(0,0) = 0.5$, and $\gamma = 0.1$.

Investors account for future speculative investments and anticipate that equilibrium prices will offer even greater speculative prices in the bad states of the world.

Proposition 4.2. A zero coupon bond which pays one unit and matures at time $T$ has price at time 0 given by

$$e^{-\gamma T} \left( \frac{\eta}{\max[\eta, 1-\eta]} \sum_{j=n^*}^{\infty} \frac{\Gamma(j + z_A)}{j! \Gamma(z_A)} \frac{k_A^{z_A}}{(T + k_A)^{j+z_A}} + \frac{1-\eta}{\max[\eta, 1-\eta]} \sum_{j=0}^{n^*-1} \frac{\Gamma(j + z_B)}{j! \Gamma(z_B)} \frac{k_B^{z_B}}{(T + k_B)^{j+z_B}} \right)$$

(4.82)

and $n^*$ is given by the smallest nonnegative integer $n$ for which

$$\eta \left( \frac{\Gamma(n + z_A)}{\Gamma(z_A)} \frac{k_A^{z_A}}{(T + k_A)^{n+z_A}} \right) > (1-\eta) \left( \frac{\Gamma(n + z_B)}{\Gamma(z_B)} \frac{k_B^{z_B}}{(T + k_B)^{n+z_B}} \right)$$

(4.83)

Figures 17 and 26 compare the term structure of interest rates when the stock price is highest, that is when $\eta = \frac{1}{2}$. In both cases we see a steeply upward sloping term structure. The standard model of term structure usually associates a steeply upward sloping term structure with economic expansion. Here we see that a steeply upward sloping term structure can also be associated with wealthy speculators who perceive profitable speculative profits in the future. The term structure is steeper for the learning case as opposed to the case where there is no learning. This is because our speculators know that each other will weight the data heavily when revising their estimates and future speculative trade will be less profitable.
5 Survival

5.1 Survival: No Learning

Now suppose the true probability measure is $P$ and under $P$, the intensity of the poisson process $N$ is given by $\lambda$ and $\lambda^A > \lambda^B$ (the opposite case is simply a relabeling exercise). From the strong law of large numbers we know $\lim_{t \to \infty} \frac{N_t^*}{t} = \lambda$, $P$ almost surely. Therefore since $\lim_{t \to \infty} \frac{n^*_t(t)}{t} = \frac{\lambda^A - \lambda^B}{\log(\lambda^A) - \log(\lambda^B)}$, we can deduce $A$ survives if

$$\frac{\lambda^A - \lambda^B}{\log(\lambda^A) - \log(\lambda^B)} < \lambda$$

(5.84)

and $B$ survives if

$$\frac{\lambda^A - \lambda^B}{\log(\lambda^A) - \log(\lambda^B)} > \lambda$$

(5.85)

For $0 < y < x$ we have the inequality

$$\sqrt{xy} < \frac{x - y}{\log(x) - \log(y)} < \frac{a + b}{2}$$

(5.86)

so a sufficient condition for $A$ to survive is

$$\frac{\lambda^A + \lambda^B}{2} < \lambda$$

(5.87)

and a sufficient condition for $B$ to survive is

$$\sqrt{\lambda^A \lambda^B} > \lambda$$

(5.88)

which leads to
**Proposition 5.1.** If $\lambda^A \neq \lambda^B$ and $\lambda^A = \lambda$ then $A$ survives and $B$ does not. If $\lambda^A \neq \lambda^B$ and $\lambda^B = \lambda$ then $B$ survives and $A$ does not. In other words the investor class with the correct beliefs always survives and in this case the investor class with the wrong beliefs never survives in a Pareto Optimal Allocation with strictly positive planner weights. However, from the individual’s perspective, they always believe they will survive and the other investor class will not, that is
\[
\lim_{t \to \infty} P^B \{ N_t \leq n^*(t) \} = 1 \quad (5.89)
\]
\[
\lim_{t \to \infty} P^A \{ N_t > n^*(t) \} = 1 \quad (5.90)
\]
The last statement follows from $\lim_{t \to \infty} N_t = \lambda^A, P^A$ almost surely and $\lim_{t \to \infty} N_t = \lambda^B, P^B$ almost surely.

Recall in the case with learning, survival depended on the wealth distribution as well as investors prior densities evaluated at the truth. In the case with no learning, as long as agents have positive wealth, regardless of the wealth distribution, the agent whose beliefs are in a sense closest to the truth survive. This is because their prior distribution is degenerate. Asymptotically, agents concentrate their wealth into these degenerate states.

**5.2 Survival: Learning**

A consumes when
\[
\eta \left( \frac{\Gamma(N_t + z_A)}{\Gamma(z_A)} \frac{k_A^{z_A}}{(t + k_A)^{N_t+z_A}} \right) > (1 - \eta) \left( \frac{\Gamma(N_t + z_B)}{\Gamma(z_B)} \frac{k_B^{z_B}}{(t + k_B)^{N_t+z_B}} \right) \quad (5.91)
\]
Rearranging
\[
\frac{\eta \frac{\Gamma(z_B)}{\Gamma(z_A)} k_A^{z_A}}{1 - \eta \frac{\Gamma(z_A)}{\Gamma(z_B)} k_B^{z_B}} > \frac{\Gamma(N_t + z_B)}{\Gamma(N_t + z_A)} \frac{(t + k_A)^{N_t+z_A}}{(t + k_B)^{N_t+z_B}} \\
= N_t^{z_A-z_B} \Gamma(N_t + z_B) \left( \frac{N_t}{t} \right)^{z_B-z_A} \left( 1 + \frac{k_A}{t} \right)^{N_t} \left( 1 + \frac{k_B}{t} \right)^{N_t} \\
= N_t^{z_A-z_B} \Gamma(N_t + z_B) \left( \frac{N_t}{t} \right)^{z_B-z_A} \exp \left( \frac{N_t}{t} \left( t \log \left( 1 + \frac{k_A}{t} \right) - t \log \left( 1 + \frac{k_B}{t} \right) \right) \right) \quad (5.92)
\]
Using Abramowitz and Stegun Equation 6.1.46
\[
\lim_{n \to \infty} n^{a-b} \frac{\Gamma(n + b)}{\Gamma(n + a)} = 1 \quad (5.93)
\]
and the fact
\[
\lim_{t \to \infty} t \log \left( 1 + \frac{k_i}{t} \right) = k_i \quad (5.94)
\]
and $\frac{N_t}{t} \to \lambda$, we see that A consumes in the limit when

$$\frac{\eta}{1 - \eta} \frac{\Gamma(z_B) k_A^{z_A}}{k_B^{z_B}} > \lambda^{z_B - z_A} \exp\left[\lambda (k_A - k_B)\right]$$

(5.95)

and B consumes in the limit when

$$\frac{\eta}{1 - \eta} \frac{\Gamma(z_B) k_A^{z_A}}{k_B^{z_B}} < \lambda^{z_B - z_A} \exp\left[\lambda (k_A - k_B)\right]$$

(5.96)

This can be rearranged to get

**Proposition 5.2.** Suppose the true intensity of the Poisson process is $\lambda$. Then A survives in the limit if

$$\frac{\eta}{1 - \eta} \frac{\Gamma(z_A)}{\Gamma(z_B)} k_A^{z_A} e^{-\lambda k_A} \lambda^{z_A - 1} > (1 - \eta) \frac{\Gamma(z_B)}{\Gamma(z_A)} k_B^{z_B} e^{-\lambda k_B} \lambda^{z_B - 1}$$

(5.97)

and investor class B survives if

$$\frac{\eta}{1 - \eta} \frac{\Gamma(z_A)}{\Gamma(z_B)} k_A^{z_A} e^{-\lambda k_A} \lambda^{z_A - 1} < (1 - \eta) \frac{\Gamma(z_B)}{\Gamma(z_A)} k_B^{z_B} e^{-\lambda k_B} \lambda^{z_B - 1}$$

(5.98)

In contrast to the results in Blume and Easley where agents satisfy Inada conditions and all Bayesian learners survive if the support of their prior contains the truth, survival of risk neutral Bayesian learners depends on the initial wealth distribution as well as their prior density evaluated at the truth. For example if $\eta = 1 - \eta$ so prices are maximized at time 0, then the agent whose density evaluated at the truth is highest survives.

### 6 Conclusion

In this paper we examined several variants of speculative models with and without learning in the framework of a simple continuous time Markov chain. Models with unlimited capital along the lines of Harrison and Kreps(1978) do not capture the effects of limited capital on asset prices. When we examine a version of this model in which interest rates are endogenous, many interesting findings arise. First, asset prices are higher than individuals private valuations when the wealth distribution is reasonably similar across individuals. The speculative premium is driven by a disagreement effect and also an interest rate effect. The interest rate effect is due to precautionary speculation – the idea that adverse shifts in states will produce even better speculative prices. We also briefly sketch out the effects of risk aversion. When investors are more risk averse than log speculative prices are below the average of individual valuations while when agents are less risk averse than log speculative prices are higher than the average of individual valuations. Many of our results can be generalized to more complex settings. The basic findings would be robust to many of these generalizations however.
References


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