Moment Problems with Applications to VaR and Portfolio Management

Ruilin Tian and Samuel H. Cox

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Part I

Bounds For Joint Extreme Events and Value-at-Risk
What we want to estimate?

\[ \Pr(X \leq x_0) \]

\[ \updownarrow \]

Confidence IV of \( \Pr(X \leq x_0) \)

\[ \updownarrow \]

100\% confidence IV of \( \Pr(X \leq x_0) \)

\[ \updownarrow \]

Bounds on \( E(I_{X \leq x_0}). \)
General form

\[
\max (\text{or } \min) \quad E[\phi(X)]
\]

where \(X\) is a set of random variables with specified support and moments.
Both objective function and constraints are in convex sets.

Example: Chebyshev’s inequality

\[
\max_X E[\phi(X)] = \frac{1}{k^2}
\]

such that \(E[X] = \mu\)

\[
E[(X - E(X))^2] = \sigma^2
\]

where \(\phi(X) = \begin{cases} 1 & \text{if } |X - \mu| > k\sigma, \\ 0 & \text{if } |X - \mu| \leq k\sigma. \end{cases}\)
Joint Probability Bounds: Bounds on $\Pr(X_1 \leq t_1 \text{ and } X_2 \leq t_2)$ for the non-negative variables $X_1$ and $X_2$ ($X_1, X_2 \geq 0$), given the mean, variance and covariance, with the objective function

$$\phi(X_1, X_2) = \mathbb{I}\{X_1 \leq t_1 \text{ and } X_2 \leq t_2\}$$

where $t_1, t_2 \in \mathbb{R}^+$. 

VaR Probability Bounds: Bounds on $\Pr(w_1 X_1 + w_2 X_2 \leq a)$ for any $X_1, X_2 \in \mathbb{R}^2$, given the mean, variance and covariance, with the objective function

$$\phi(X_1, X_2) = \mathbb{I}\{w_1 X_1 + w_2 X_2 \leq a\}$$

where $w_1, w_2, a \in \mathbb{R}$. 

Bounds on Stop-loss payments

\[
\phi(X_1, X_2) = \begin{cases} 
  b & \text{if } X_1 + X_2 \geq a + b \\
  X_1 + X_2 - a & \text{if } a \leq X_1 + X_2 \leq a + b \\
  0 & \text{if } X_1 + X_2 \leq a.
\end{cases}
\]

- Special cases: payoffs of call or put options
- This problem can be converted to a one variable moment problem and solved numerically.
- We also can use the explicit formulae deduced by Cox (1991) to compute the bounds.
In order to numerically solve the semiparametric bounds, we reformulate the corresponding semiparametric bound problem as a sum of squares (SOS) program using the following two theories:

▶ **Theorem (Hilbert(1888))**

Let \( p(x_1, \ldots, x_n) \) be a quadratic polynomial. Then \( p(x_1, \ldots, x_n) \geq 0, \ \forall \ x_1, \ldots, x_n \in \mathbb{R} \) if and only if \( p(x_1, \ldots, x_n) \) is a SOS polynomial.

▶ **Theorem (Diananda(1962))**

Let \( p(x_1, \ldots, x_n) \) be a quadratic polynomial. If \( n \leq 3 \), then \( p(x_1, \ldots, x_n) \geq 0, \ \forall \ x_1, \ldots, x_n \geq 0 \) if and only if \( p(x_1^2, \ldots, x_n^2) \) is a SOS polynomial.
A polynomial

\[ p(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_n \in \mathbb{N}} y(i_1, \ldots, i_n) x_1^{i_1} \cdots x_n^{i_n} \]

is said to be a SOS polynomial if

\[ p(x_1, \ldots, x_n) = \sum_i q_i(x_1, \ldots, x_n)^2 \]

for some polynomials \( q_i(x_1, \ldots, x_n) \).

A SOS program is an optimization problem where the variables are coefficients of polynomials, the objective is a linear combination of the variable coefficients, and the constraints are given by the polynomials being SOS polynomial.
SOS programming solvers such as SOSTOOLS (Prajna et al. (2002)), GloptiPoly (Henrion and Lasserre (2003)), or YALMIP (Löfberg(2004)) can numerically solve the SOS program.

SOS programming solvers reformulate the SOS program as a SDP and use SDP solvers such as SeDuMi (Sturm (1999)) to solve the problem. SDP formulations of SOS programs are typically involved.
Primal and Dual Problems

▶ Primal problem

\[
\overline{p}(\pi) = \max(\min) \quad \mathbb{E}_\pi(\phi(X_1, X_2))
\]

such that

\[
\begin{align*}
\mathbb{E}_\pi(1) &= 1, \\
\mathbb{E}_\pi(X_i) &= \mu_i, \quad i = 1, 2, \\
\mathbb{E}_\pi(X_i^2) &= \mu_i^{(2)}, \quad i = 1, 2, \\
\mathbb{E}_\pi(X_1X_2) &= \mu_{12},
\end{align*}
\]

\(\pi \) a probability distribution in \(\mathcal{D}\)

▶ Dual problem [Popescu (2005)]

\[
\overline{d}(\pi) = \min(\max) y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12}
\]

such that

\[
p(x_1, x_2) \geq (\text{or} \leq) \phi(x_1, x_2), \quad \forall (x_1, x_2) \in \mathcal{D},
\]

\(2\)

where the quadratic polynomial

\[
p(x_1, x_2) = y_{00} + y_{10}x_1 + y_{01}x_2 + y_{20}x_1^2 + y_{02}x_2^2 + y_{11}x_1x_2.
\]
Feasibility and Strong Duality

- **Feasibility**: The dual problem (2) is feasible if and only if $\Sigma$ is a positive semidefinite matrix (i.e., all eigenvalues are greater than or equal to zero), where $\Sigma$ is the moment matrix. If $\mathcal{D} = \mathbb{R}^{+2}$, all elements of $\Sigma$ are required to be non-negative.

$$
\Sigma = \begin{bmatrix}
1 & \mu_1 & \mu_2 \\
\mu_1 & \mu_1^{(2)} & \mu_{12} \\
\mu_2 & \mu_{12} & \mu_2^{(2)} \\
\end{bmatrix}.
$$

- **Strong Duality** The solution to the dual is equivalent to the primal in the sense that the numerical values of the dual is equal to that of the primary, if and only if one can find a special case in which the constraints in the dual is strictly satisfied.
Joint Probability Bounds on $\Pr(X_1 \leq t_1 \text{ and } X_2 \leq t_2)$

- **Upper Bounds**

$$
\bar{d} = \min \ y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12}
$$

such that

- $p(x_1, x_2) \geq 1, \forall \ 0 \leq x_1 \leq t_1, 0 \leq x_2 \leq t_2$
- $p(x_1, x_2) \geq 0, \forall \ x_1, x_2 \geq 0.$

- **Lower Bounds**

$$
\underline{d} = \max \ y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12}
$$

such that

- $p(x_1, x_2) \leq 1, \forall \ 0 \leq x_1 \leq t_1, 0 \leq x_2 \leq t_2$
- $p(x_1, x_2) \leq 0, \forall \ x_1, x_2 \geq 0.$
Numerical Example of Probability Bounds

Consider the American International Group (AIG). We find bounds on the probability of joint extreme events, i.e., unanticipated poor asset returns and unexpectedly high claims, \( \Pr(r \leq t_1, m \leq t_2) \).

**X₁: Weighted Average Net Return.** Suppose AIG’s portfolio contains 6 assets. For asset \( i \), \( r_i^G = r_i + 1 = \frac{p_{i,t}}{p_{i,t-1}} \).

\( r \) is the weighted average return:

\[
r = \sum_{i=1}^{6} w_i r_i^G - 1.
\]

**X₂: Margin.** The margin on insurance business \( m \) is defined as

\[
m = 1 - LR,
\]

where \( LR \) is the economic loss ratio.
The upper left plot shows the upper bound of $\Pr(r \leq t_1, m \leq t_2)$. The upper right one is the CDF of bivariate normal with the same moments as AIG. The ratio of the upper bound to the bivariate normal CDF is shown in the third graph. The vertical axis of the graphs is probability. It is the ratio in the third graph.
The higher curve is the upper bound on $\Pr(r \leq t_1, m \leq t_2)$. The lower curve is the CDF of bivariate normal with the same moments as AIG. x-axis stands for $t_1$. And $t_2$ is fixed at $E(m) - k\sigma(m)$ where $k = 0.25, 0.5, \ldots, 1.5$, with $k = 0.25$ on the upper left and running to the right and then down.
VaR Bounds on $\Pr(w_1X_1 + w_2X_2 \leq a)$

### Upper Bounds

$$d_{\text{VaR}} = \min \ y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12}$$

such that

$$p(x_1, x_2) \geq 1, \forall \ x_1, x_2 \text{ s.t. } w_1x_1 + w_2x_2 \leq a$$

$$p(x_1, x_2) \geq 0, \forall \ x_1, x_2 \in \mathbb{R}.$$ 

### Lower Bounds

$$d_{\text{VaR}} = \max \ y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12}$$

such that

$$p(x_1, x_2) \leq 1, \forall \ x_1, x_2 \text{ s.t. } w_1x_1 + w_2x_2 \leq a$$

$$p(x_1, x_2) \leq 0, \forall \ x_1, x_2 \in \mathbb{R}.$$
Numerical Example of VaR Bounds

▶ We analyze the tail joint probability of total return (annual return) of a portfolio investing in the S&P500 and Nikkei indices.

\[
\begin{align*}
E(r_{sp}) &= 0.1107 \\
E(r_{sp}^2) &= 0.0349 \\
E(r_{nk}) &= 0.0473 \\
E(r_{nk}^2) &= 0.0554 \\
Cov(r_{sp}, r_{nk}) &= 0.0145
\end{align*}
\]

These two indices are correlated with \( \rho = 0.4190 \).

▶ Suppose we invest 50% of our funding in the S&P500 and 50% in Nikkei.
Bounds for the probability $\Pr(0.5r_{sp} + 0.5r_{nk} \leq a)$. The vertical axis is the probability and the horizontal axis stands for different values of $a$. The 5% VaR using the normal distribution is $a_{Normal} = -0.20$. The semiparametric bounds are $a_L < \text{VaR}_{0.05} < a_H$ with $a_L = -0.7$ and $a_H = 0.1$. 
Conclusion

- Analyze two bound problems on bivariate distributions
  - “100% confidence intervals” on extreme events given moments and support
  - Bounds on the sum of two variables given moments and support

- Other possible applications of our approach
  - Default probabilities
  - Prices of different fix-income securities
  - Inventory and supply chain management
  - Sensitivity analysis of the joint probabilities and VaR estimates to model misspecification
Future Work

▶ **Robustness test**: sensitivity analysis of bounds estimations with respect to changes of moments

▶ **Dynamic bounds estimation**: update moment information over time (rolling)

▶ **Accuracy improvement**: add additional assumptions such as unimodal distribution or symmetric distribution (narrow the 100% confidence interval)
Portfolio Risk Management with CVaR-Like Constraints
What we want to do?
Conditional Tail Expectation

- $\beta$-level VaR:

$$\alpha(x, \beta) = \min\{\alpha \in \mathbb{R} : \mathbb{P}(R(x) \leq \alpha) \geq \beta\}.$$  

- $\beta$-level CVaR:

$$\text{CVaR}(x, \beta) = \mathbb{E}(R(x) | R(x) \leq \alpha(x, \beta)).$$

- Based on Rockefeller and Uryasev (2000), the left-tail $\beta$-level CVaR can be derived as follows:

$$\text{CVaR}(x, \beta) = \max_{\alpha} \alpha - \frac{1}{\beta} \mathbb{E}((\alpha - R(x))^+).$$

Where $(x)^+$ is defined as $\max(x, 0)$. 

Term Definitions

Assume: \( R_i \) is the return of asset \( i \). \( r_{ij} \) is the observed value of \( R_i \) in year \( j \), for \( i = 1, \ldots, n \) (assets) and \( j = 1, \ldots, m \) (observations).

- Model distribution moments
  \[
  \mu_i = \mathbb{E}[R_i], \quad i = 1, \ldots, n.
  \]
  \[
  \sigma_{ij} = \mathbb{E}[(R_i - \mu_i)(R_j - \mu_j)], \quad i, j = 1, \ldots, n.
  \]
  \[
  \gamma_{ijk} = \mathbb{E}[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)], \quad i, j, k = 1, \ldots, n.
  \]

- Empirical distribution moments
  \[
  \hat{\mu}_i = \frac{1}{m} \sum_{l=1}^{m} r_{il}, \quad i = 1, \ldots, n.
  \]
  \[
  \hat{\sigma}_{ij} = \frac{1}{m} \sum_{l=1}^{m} (r_{il} - \hat{\mu}_i)(r_{jl} - \hat{\mu}_j), \quad i, j = 1, \ldots, n.
  \]
  \[
  \hat{\gamma}_{ijk} = \frac{1}{m} \sum_{l=1}^{m} (r_{il} - \hat{\mu}_i)(r_{jl} - \hat{\mu}_j)(r_{kl} - \hat{\mu}_k), \quad i, j, k = 1, \ldots, n.
  \]
Portfolio Moments

- Weights vector \( X = [x_1, x_2, \ldots, x_n] \)
- Portfolio empirical return in year \( j \):

\[
\hat{\mu}(x)_j = \sum_{i=1}^{n} r_{ij} x_i \quad \forall \ j = 1, \ldots, m.
\]

- Empirical moments of portfolio:

\[
\begin{align*}
\hat{\mu}(x) &= \frac{1}{m} \sum_{l=1}^{m} \hat{\mu}(x)_l = \frac{1}{m} \sum_{l=1}^{m} \sum_{i=1}^{n} r_{il} x_i = \sum_{i=1}^{n} \hat{\mu}_i x_i, \\
\hat{\sigma}^2(x) &= \frac{1}{m} \sum_{l=1}^{m} [\hat{\mu}(x)_l - \hat{\mu}(x)]^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{\sigma}_{ij} x_i x_j, \\
\frac{1}{m} \sum_{l=1}^{m} [\hat{\mu}(x)_l - \hat{\mu}(x)]^3 &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \hat{\gamma}_{ijk} x_i x_j x_k.
\end{align*}
\]
Optimization with CVaR Constraint

Minimize \[
\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j
\]

subject to \[
\text{CVaR}(x, \beta) \geq w
\]

\[
\sum_{i=1}^{n} x_i = 1
\]

\[
\sum_{i=1}^{n} \mu_i x_i = \mu_0(x)
\]

\[
x_i \geq 0, \quad i = 1, 2, \ldots, n.
\]
Transformation of CVaR Constraint

$$\text{CVaR}(x, \beta) \geq w$$

$$\implies$$

$$\max_{\alpha} \quad \alpha - \frac{1}{\beta} \mathbb{E}((\alpha - R(x))^+) \geq w$$

$$\implies$$

$$\max_{\alpha} \quad \alpha - \frac{1}{\beta} \sum_{j=1}^{m} y_j \geq w$$

subject to

$$y_j \geq \alpha - \sum_{i=1}^{n} r_{ij} x_i, \quad j = 1, \ldots, m$$

$$y_j \geq 0, \quad j = 1, \ldots, m.$$

Linearization process follows Konno and Yamazaki(1991)'s technique to $$(.)^+$$.
Optimization with CVaR-Like Constraint

Minimize

\[\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j\]

subject to

\[\alpha - \frac{1}{\beta} \frac{1}{m} \sum_{j=1}^{m} y_j \geq w\]

\[y_j \geq \alpha - \sum_{i=1}^{n} r_{ij} x_i, \quad j = 1, \ldots, m\]

\[\sum_{i=1}^{n} x_i = 1\]

\[\sum_{i=1}^{n} \mu_i x_i = \mu_0(x)\]

\[x_i, y_j \geq 0 \quad i = 1, 2, \ldots, n; j = 1, 2, \ldots, m.\]
Optimization with 5%-CVaR-like constraint:
Reshape distribution with $p$ CVaR-like constraints

More than one CVaR-like constraints can be added to the optimization problem to reshape different quantiles.
Optimization with $p$ CVaR-Like Constraints

Quantiles $\beta^1, \beta^2, \ldots, \beta^p \in (0, 1)$ and $w^1, w^2, \ldots, w^p \in \mathbb{R}$:

Minimize

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j$$

subject to

$$\alpha^l - \frac{1}{\beta^l} \frac{1}{m} \sum_{j=1}^{m} y^l_j \geq w^l, \quad l = 1, 2, \ldots, p$$

$$y^l_j \geq \alpha^l - \sum_{i=1}^{n} r_{ij} x_i, \quad j = 1, 2, \ldots, m; \quad l = 1, 2, \ldots, p$$

$$\sum_{i=1}^{n} x_i = 1; \quad \sum_{i=1}^{n} \mu_i x_i = \mu_0(x)$$

$$x_i \geq 0, \quad i = 1, 2, \ldots, n$$

$$y^l_j \geq 0, \quad j = 1, 2, \ldots, m; \quad l = 1, 2, \ldots, p$$

$$\alpha^l \in \mathbb{R}, \quad l = 1, 2, \ldots, p.$$
Use CVaR to measure risks and maximize the worst case.

Maximize \[ \text{CVaR}(x, \beta) \]

subject to \[ \sum_{i=1}^{n} x_i = 1 \]

\[ \sum_{i=1}^{n} \mu_i x_i = \mu_0(x) \]

\[ x_i \geq 0, \quad i = 1, \ldots, n. \]

Again, apply the technique of Konno and Yamazaki (1991) to linearize the problem.
Alternative 2: Mean-Absolute Deviation (MAD) Approach

\[
\text{MAD}(R(x)) = \frac{1}{m} \sum_{j=1}^{m} \left| \sum_{i=1}^{n} r_{ij} x_i - \sum_{i=1}^{n} \hat{\mu}_i x_i \right| = \frac{1}{m} \sum_{j=1}^{m} \left| \sum_{i=1}^{n} (r_{ij} - \hat{\mu}_i) x_i \right|
\]

Minimize \[\frac{1}{m} \sum_{j=1}^{m} y_j\]

subject to \[y_j \geq \sum_{i=1}^{n} (r_{ij} - \hat{\mu}_i) x_i, \quad j = 1, \ldots, m\]

\[y_j \geq - \sum_{i=1}^{n} (r_{ij} - \hat{\mu}_i) x_i, \quad j = 1, \ldots, m\]

\[\sum_{i=1}^{n} x_i = 1; \quad \sum_{i=1}^{n} \hat{\mu}_i x_i = \mu_0(x)\]

\[x_i, y_j \geq 0, \quad i = 1, \ldots, n; j = 1, \ldots, m.\]
Boyle and Ding (2006) give a method to increase the skewness of a given portfolio $x^*$, obtaining a new portfolio $x$ for which the mean returns are equal and the variance of returns may increase at most by $\epsilon$. Then the skewness of the new portfolio $R(x)$ should be greater than the skewness of the original $R(x^*)$.

\[
\begin{align*}
\mu(x) &= \mu(x^*) \\
\sigma^2(x) &\geq \sigma^2(x^*) + \epsilon \\
\frac{1}{m} \sum_{j=1}^{m} [\mu(x)_j - \mu(x)]^3 &\geq \frac{1}{m} \sum_{j=1}^{m} [\mu(x^*)_j - \mu(x^*)]^3 + \delta,
\end{align*}
\]

where both $\epsilon$ and $\delta$ are small positive numbers.
When adding more assets or considering asset-liability portfolios, we get similar outcomes.
Asset-liability portfolio: insurer’s portfolio taking into account both the asset investment and claim liabilities.

- Insurers collect premium at the beginning of the year for several lines of business and also allocate additional capital in revenues expressed by $\lambda_i$. For each line of business, $\Pi_i(1 + \lambda_i)$ is invested.

$$\sum_{i=1}^{k_1} \Pi_i(1 + \lambda_i) = \sum_{i=1}^{k_1} \Pi_i + \sum_{i=1}^{k_1} \Pi_i \lambda_i = \Pi(1 + \lambda).$$

- Insurers pay out losses and expenses at the end of the year.

$$\text{Margin} = \frac{\text{Written premiums} - \text{Losses Incurred} - \text{Expenses}}{\text{Written premiums}}$$
Consider an insurer operates in \( k_1 \) lines of business and invests in \( k_2 \) assets. The total profits in the company’s favor at the end of the year are written as follows:

\[
(1 + \lambda)\prod_{i=1}^{k_1} \left( \sum_{i=1}^{k_1} a_i M_i + \sum_{j=1}^{k_2} b_j R^*_j + \lambda \right),
\]

where \( L_i \) is the sum of claim payments and administrative expenses in year \( i \), \( M_i = 1 - \frac{L_i}{\prod_i} \) and \( R^*_j = (1 + \lambda)R_j \).

Therefore, in the optimization problem, we have two constraints about weights:

\[
a_1 + a_2 + \cdots + a_{k_1} = 1
\]

\[
b_1 + b_2 + \cdots + b_{k_2} = 1
\]
Conclusions

▶ Develop a new effective way to improve the skewness of the mean-variance portfolio — CVaR-like constraints approach.
▶ Apply CVaR-like constraints approach to asset-liability portfolio.
▶ Compare the CVaR-like constraints frontier with Markowitz (1952)’s mean-variance (MV) and Boyle-Ding (2006)’s mean-variance-skewness (MVS) frontiers as well as those based on other risk measures such as mean-absolute deviation (MAD) and CVaR itself.
▶ Empirical analysis shows CVaR-like constraints approach is the most effective way to improve the skewness of mean-variance portfolios among the five methods analyzed.