The cost of counterparty risk and collateralization in longevity swaps∗

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Abstract

Derivative longevity risk solutions, such as bespoke and indexed longevity swaps, allow pension schemes and annuity providers to swap out of longevity risk, but introduce counterparty credit risk, which can be mitigated if not fully eliminated by collateralization. In this study, we examine the impact of bilateral default risk and collateral flows on the marking to market of longevity swaps, and show how different rules for posting collateral during the life of these instruments can affect longevity swap rates. For typical interest rate and mortality parameters, we find that the cost of collateral is modest in the presence of symmetric default risk, but more pronounced when default risk and/or collateral rules are asymmetric. Our results suggest that the overall cost of collateralization is comparable with that documented for the interest-rate swaps market.

Keywords: longevity swap, counterparty risk, default risk, collateral, marking-to-market.

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1 Introduction

The market for longevity-linked securities and derivatives has recently experienced a surge in transactions in longevity swaps. These are agreements between two parties to exchange fixed payments against variable payments linked to the number of survivors in a reference population (see Dowd et al., 2006). Table 1 presents a list of recent deals that have been publicly disclosed. So far, transactions have mainly involved pension funds and annuity providers wanting to hedge their exposure to longevity risk but without having to bear any basis risk: the variable payments in such longevity swaps are designed to match precisely the mortality experience of each individual hedger (hence the name indemnity-based, or bespoke, longevity swaps). This is essentially a form of longevity risk insurance, similar to annuity reinsurance in reinsurance markets. A fundamental difference, however, is that longevity swaps are typically collateralized, whereas insurance/reinsurance transactions are not.\(^{1}\) The main reason is that longevity swaps are often part of a wider de-risking strategy involving other collateralized instruments (interest-rate and inflation swaps, for example), and also the fact that hedgers have been increasingly concerned with counterparty risk\(^{2}\) in the wake of the Global Financial Crisis. In this article, we provide a framework to quantify the trade-off between the exposure to counterparty risk in longevity swaps and the cost of credit enhancement strategies such as collateralization.

As there is no accepted framework yet for marking to market/model longevity swaps, hedgers and hedge suppliers look to other markets to provide a reference model for counterparty risk assessment and mitigation. In interest-rate swap markets, for example, the most common form of credit enhancement is the posting of collateral. According to

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\(^{1}\)One rationale for this is that reinsurers aggregate several uncorrelated risks and pooling/diversification benefits compensate for the absence of collateral (e.g., Lakdawalla and Zanjani, 2007; Cummins and Trainar, 2009).

\(^{2}\)Basel II (2006, Annex 4) defines counterparty risk as “the risk that the counterparty to a transaction could default before the final settlement of the transaction’s cash flows”. The recent Solvency II proposal makes explicit allowance for a counterparty risk module in its ‘standard formula’ approach; see CEIOPS (2009).
the International Swap and Derivatives Association (ISDA) almost every swap at major financial institutions is ‘bilaterally’ collateralized (ISDA, 2010b), meaning that either party is required to post collateral depending on whether the market value of the swap is positive or negative.\(^3\) The vast majority of transactions is collateralized according to the Credit Support Annex to the Master Swap Agreement introduced by ISDA (1994). The Global Financial Crisis of 2008-09 highlighted the importance of bilateral counterparty risk and collateralization for over-the-counter markets, spurring a number of responses (e.g., ISDA, 2009; Brigo and Capponi, 2009; Assefa et al., 2010; Brigo et al., 2011). The Dodd-Frank Wall Street Reform and Consumer Protection Act (signed into law by President Barack Obama on July 21, 2010) is likely to have a major impact on the way financial institutions will manage counterparty risk in the coming years.\(^4\) The recently founded Life and Longevity Markets Association\(^5\) (LLMA) has counterparty risk at the center of its agenda, and will certainly draw extensively from the experience garnered in fixed-income and credit markets.

The design of collateralization strategies is intended to address the concerns aired by pension trustees regarding the efficacy of longevity swaps, but introduces another dimension in the traditional pricing framework used for insurance transactions. The ‘insurance premium’ embedded in a longevity swap rate reflects not only the aversion (if any) of the counterparties to the risk being transferred and the cost of regulatory capital involved in the transaction, but also the expected costs to be incurred from posting collateral during the life of the swap. The fact that collateral is costly simply reflects the costs entailed by credit risk mitigation. To sign and gauge the impact of collateral on swap rates, we must examine the relative sensitivity of the counterparties

\(^3\)Unlike a firm’s exposure to credit risk through a loan, where the exposure to credit risk is unilateral and only the lending bank faces the risk of loss, counterparty credit risk creates a bilateral risk of loss: the market value of the transaction can be positive or negative to either counterparty to the transaction. The market value is uncertain and can vary over time with the movement of underlying market factors.” (Basel II, 2006, Annex 4).

\(^4\)See, for example, “Berkshire may scale back derivative sales after Dodd-Frank”, Bloomberg, August 10, 2010.

to its cost. Let us first take the perspective of a hedge supplier (reinsurer or investment bank) issuing a collateralized longevity swap to a counterparty (pension fund or annuity provider). Whenever the swap is sufficiently out-of-the-money, the hedge supplier is required to post collateral, which can be used by the hedger to mitigate losses in the event of default. Although interest on collateral is (partially) rebated, there is an opportunity cost, as the posting of collateral depletes the resources the hedge supplier can use to meet her capital requirements at aggregate level. On the other hand, whenever the swap is sufficiently in-the-money, the hedge supplier will receive collateral from the counterparty, thus benefiting from capital relief in regulatory valuations. The benefits can be far larger if collateral can be re-pledged for other purposes, as in the interest-rate swaps market.\footnote{According to ISDA (2010b), the vast majority of collateral is rehypothecated for other purposes in interest-rate swap markets. Currently, collateral can be re-pledged under the New York Credit Support Annex, but not under the English Credit Support Deed (see ISDA, 2010a).}

The same considerations can be made from the viewpoint of the hedger, but the funding needs of the two parties are unlikely to offset each other exactly. This is particularly relevant for transactions involving parties subject to different regulatory frameworks. In the UK and several other countries, for example, longevity risk exposures are more capital intensive for hedge suppliers than for pension funds.\footnote{This asymmetry is a by-product of rules allowing, for example, liabilities to be quantified by using outdated mortality tables or discount rates reflecting optimistic expected returns.}

In the absence of collateral, and ignoring longevity risk aversion, swap rates depend on best estimate survival probabilities for the hedged population and on the degree of covariation between the floating leg of the swap and the defaultable term structure of interest rates facing the hedger and the hedge supplier.\footnote{Along the same lines, Inkmann and Blake (2010) show how the discount rate for the valuation of pension liabilities should reflect funding risk.} This means that a proper analysis of a longevity swap cannot disregard the sponsor’s covenant when the hedger is a pension plan (see section 3 below). In the presence of collateralization, longevity swap rates are also shaped by the expected collateral costs, and swap valuation formulae involve a discount rate reflecting the opportunity cost of collateral. This means, in
particular, that default-free valuation formulae are not appropriate even in the presence of full collateralization and the corresponding absence of default losses (see Johannes and Sundaresan, 2007, for the case of symmetric default risk and collateral costs in interest-rate swaps).

For typical interest rate and mortality parameters, we find that the impact of collateralization on swap rates is modest when default risk and collateral rules are symmetric. There are two opposing effects at play here:

i) On the one hand, the receiver of the fixed survival rate (the hedge supplier) posts collateral when mortality is lower and hence longevity exposures are more capital intensive. On the other hand, she receives collateral when mortality is higher and longevity protection less capital intensive. The overall effect is to push swap rates higher, to compensate the hedge supplier for the positive dependence between collateral flows and capital charges.

ii) When the hedge supplier is out-of-the-money, collateral outflows are larger in low interest rate environments (i.e., when liabilities are discounted at a lower rate), hence there is a negative relationship between the amount of collateral posted and the hedge supplier’s funding costs. On the other hand, when the hedge supplier is in-the-money, collateral inflows are larger exactly when funding costs are higher. The opposite is true for the hedger who demands lower longevity swap rates as a compensation for the positive dependence between its funding costs and collateral amounts.

When default risk or collateral rules are asymmetric, either of the two effects prevail and the impact on longevity swap rates is larger. For example, we find that swap rates increase substantially when the hedger has lower credit standing and collateral rules are more favorable to the hedge supplier. Although collateralization introduces an explicit link between the individual risk exposures and the hedge supplier’s funding risk (hence some of the pooling/diversification benefits used to substitute for collateralization in the standard insurance model may be lost), in our examples we find that the overall impact
of collateralization is comparable with, and typically lower than, that observed in fixed-income markets (e.g., Johannes and Sundaresan, 2007). This means that the interest-rate swaps market might provide the right framework for the collateralization of indemnity-based solutions, even though the latter lack of the transparency and standardization benefits associated with indexed-based instruments.

On the methodological side, we show how longevity swap rates must be determined endogenously from the dynamic marking to market\(^9\) of the swap and the collateral rules specified by the contract. To see why, note that the market value of the swap at each valuation date depends on the evolution of the relevant state variables (mortality, interest rates, credit spreads), as well as on the swap rate locked in at inception. On the other hand, the swap’s market value will typically affect collateral amounts and, in a setting where collateral is costly, will embed the market value of expected future collateral flows. Hence, the swap rate can only be determined by explicitly taking into account the marking-to-market process and the dynamics of collateral posting. To avoid the computational burden of nested Monte Carlo simulations, we use an iterative procedure based on the Least-Squares Monte Carlo approach\(^{10}\) (see Glassermann, 2004, and references therein). We provide several numerical examples showing how different collateralization rules shape longevity swap rates giving rise to margins in (best estimate) survival probabilities reflecting the opportunity cost of future collateral flows. Although our focus is on longevity risk solutions, the approach can be applied to other instruments, in particular to bespoke solutions for other risks, such as inflation and credit risk.

Our work contributes to the existing literature on longevity risk pricing in at least three ways: i) we introduce default risk in the pricing of longevity risk solutions, and properly address its bilateral nature; ii) we explicitly allow for collateralization rules, which are the backbone of any real-world hedging solution and materially affect the

\(^9\)Here and in the following, by ‘market value’ and ‘marking to market’ we mean that market-consistent values are computed according to the relevant accounting/regulatory standard.

\(^{10}\)A similar approach is used by Bacchello et al. (2010) for surrender guarantees in life policies and by Bauer et al. (2010b) for the computation of capital requirements within the Solvency II framework.
pricing of over-the-counter transactions; and iii) we introduce a ‘structural’ dimension in an otherwise reduced-form pricing framework, by allowing for funding costs associated with longevity risk exposures held by hedgers and hedge suppliers. As there is essentially no publicly available information on swap rates, our approach\footnote{Similarly, Biffis and Blake (2010a) endogenize longevity risk premia by introducing asymmetric information and capital requirements in a risk-neutral setting.} has the advantage of using publicly available information on credit markets and regulatory standards, without having to rely on calibration to primary insurance market prices, approximate hedging methods or assumptions on agents’ risk preferences (e.g., Dowd et al., 2006; Ludkovski and Young, 2008; Bauer et al., 2010a; Biffis et al., 2010; Chen and Cummins, 2010; Cox et al., 2010, among others).

The article is organized as follows. In the next section, we introduce longevity swaps and formalize their payoffs. We examine the case of both indemnity-based and index-based swaps, but, in the latter case, we ignore the issue of basis risk\footnote{See, for example, Coughlan et al. (2011), Salhi and Loisel (2010) and Stevens, De Waegenaere and Melenberg (2010b) for some results related to this risk dimension.} to keep the article focused. In section 2.1, we examine the marking to market of a longevity swap during its life time to demonstrate the impact of counterparty risk on the hedger’s balance sheet. Section 3 introduces bilateral default risk in longevity swap valuation formulae. We identify the main channels through which default risk affects the market value of swaps and show why an iterative procedure is needed to compute swap rates. Section 4 introduces credit enhancement in the form of collateralization, and shows how longevity swap rates are affected even in the presence of full cash collateralization (and hence absence of default losses). We explain how swap rates can be computed by using an iterative procedure based on the Least-Squares Monte Carlo approach. In section 5, several stylized examples are provided to understand how different collateralization rules may affect longevity swap rates. Concluding remarks are offered in section 6. Further details and technical remarks are collected in an appendix.
2 Longevity swaps

We consider a hedger (insurer, pension fund), referred to as party A, and a hedge supplier (reinsurer, investment bank), referred to as counterparty B. Agent A has the obligation to pay amounts \( X_{T_1}, X_{T_2}, \ldots \), possibly dependent on interest rates and inflation, to each survivor at fixed dates \( 0 < T_1 \leq T_2, \ldots \) of an initial population of \( n \) individuals alive at time zero (annuitants or pensioners). We are clearly restricting our attention to homogeneous liabilities for ease of exposition, more general situations requiring obvious modifications. Party A’s liability at a generic payment date \( T > 0 \) is given by the random variable \((n - N_T)X_T\), where \( N_T \) counts the number of deaths experienced by the population during the period \([0, T]\). Assuming that the individuals’ death times have common intensity\(^{13}\) \((\mu_t)_{t \geq 0}\), the expected number of survivors at time \( T \) can be written as \(E^{\mathbb{P}}[n - N_T] = np_T\), with the survival probability \( p_T \) given by (see the appendix)

\[
p_T := E^{\mathbb{P}} \left[ \exp \left( - \int_0^T \mu_t dt \right) \right]. \tag{2.1}
\]

Here and in the following, \( \mathbb{P} \) denotes the real-world probability measure. The intensity could be modeled by using, for example, any of the stochastic mortality models considered in Cairns et al. (2009). For our examples, we will rely on the simple Lee-Carter model.

Let us now consider a financial market and introduce the risk-free rate process \((r_t)_{t \geq 0}\). We assume that a market-consistent price of the liabilities can be computed by using a risk-neutral measure \( \tilde{\mathbb{P}} \), equivalent to \( \mathbb{P} \), such that the death times have the same intensity process \((\mu_t)_{t \geq 0}\) (with different dynamics, in general, under the two measures; \(^{13}\)Intuitively, \( \mu_t \) represents the instantaneous conditional death probability for an individual alive at time \( t \).)
see Biffis et al., 2010). The time-0 market value of the aggregate liability can then be written as

\[ E^{\mathbb{P}} \left[ \sum_i \exp \left( -\int_0^{T_i} r_t \, dt \right) (n - N_{T_i}) X_{T_i} \right] = n \sum_i E^{\mathbb{P}} \left[ \exp \left( -\int_0^{T_i} (r_t + \mu_t) \, dt \right) X_{T_i} \right]. \]

For the moment, we take the pricing measure as given: we will give it more structure later on.

We consider two instruments which A can enter into with B to hedge its exposure: an indemnity-based longevity swap and an index-based longevity swap. In these swaps, in contrast with interest rate swaps, the swap rate needs to be a collection of fixed rates pertaining to each individual payment date. The reason is that mortality increases substantially at old ages and a fixed rate would introduce a growing mismatch between the cashflows provided by the swap and those needed by the hedger. However, as with interest rate swaps, we can treat a longevity swap as a portfolio of forward contracts on the underlying floating (survival) rate.\(^{14}\) In this section, we ignore default risk and focus on individual payments at maturity \( T > 0 \). In what follows, we always assume the perspective of the hedger.

An indemnity-based longevity swap allows party A to pay a fixed rate \( \bar{p}^N \in (0, 1) \) against the realized survival rate experienced by the population between time zero and time \( T \). Assuming a notional amount equal to the initial population size, \( n \), the net payout to the hedger at time \( T \) is\(^{15}\)

\[ n \left( \frac{n - N_T}{n} - \bar{p}^N \right), \]

\(^{14}\)With a slight abuse of terminology, we use the term ‘swap rate’ for individual forward rates as well as for swap curves (collection of swap rates). We note that swap curves are often summarized by the improvement factor applied to the survival probabilities of a reference mortality table/model.

\(^{15}\)For ease of exposition, here and in the following sections, we consider contemporaneous settlement only. Other settlement conventions (e.g. in arrears) have negligible effects, but make valuation formulae more involved when bilateral and asymmetric default risk is introduced.
i.e., the difference between the realized number of survivors and the pre-set number of survivors \( n \tilde{p}^N \) agreed at inception. Letting \( S_0 \) denote the market value of the swap at inception, we can write

\[
S_0 = n E_{\tilde{P}} \left[ \exp \left( - \int_0^T r_t dt \right) \left( \frac{n - N_T}{n} - \tilde{p}^N \right) \right] \\
= n E_{\tilde{P}} \left[ \exp \left( - \int_0^T (r_t + \mu_t) dt \right) \right] - n B(0, T) \tilde{p}^N, \tag{2.2}
\]

with \( B(0, T) \) denoting the time-zero price of a zero-coupon bond with maturity \( T \). By setting \( S_0 = 0 \), we obtain the swap rate as

\[
\tilde{p}^N = \tilde{p}_T + B(0, T)^{-1} \text{Cov}_{\tilde{P}} \left( \exp \left( - \int_0^T r_t dt \right), \exp \left( - \int_0^T \mu_t dt \right) \right), \tag{2.3}
\]

where the risk-adjusted survival probability \( \tilde{p}_T \) is defined as in (2.1) by taking expectations under \( \tilde{P} \). Expression (2.3) shows that if the intensity of mortality is uncorrelated with bond market returns (a reasonable first-order approximation), the longevity swap curve just involves the survival probabilities \( \{ \tilde{p}_T_i \} \) relative to the different maturities \( \{ T_i \} \). Several studies have recently addressed the issue of how to quantify risk-adjusted survival probabilities, for example, by calibration to annuity prices and books of policies traded in secondary markets, or by use of approximate hedging methods (see references in Section 1). As there is essentially no publicly available information on swap rates, we will suppose a baseline case in which \( \tilde{p}_T_i = p_T_i \) for each maturity \( T_i \) and focus on how counterparty default risk and collateral requirements might shape longevity swap rates. Although in what follows, we mainly concentrate on longevity risk, in practice, the floating payment of a longevity swap may involve a LIBOR component or survival indexation rules different from the ones considered above. To keep the setup general, we will at times consider instruments making a generic variable payment, \( P \), and write the
corresponding swap rate $p$ as

$$p = E^{\tilde{\mathbb{B}}} [P] + B(0,T)^{-1} \text{Cov}^{\tilde{\mathbb{B}}} \left( \exp \left( -\int_0^T r_t dt \right), P \right). \quad (2.4)$$

The setup can easily accommodate **index-based longevity swaps**, standardized instruments allowing the hedger to pay a fixed rate $\bar{p}^I \in (0,1)$ against the realized value of a survival index $(I_t)_{t \geq 0}$ at time $T$. The latter might reflect the mortality experience of a reference population closely matching that of the liability portfolio. Examples are represented by the LifeMetrics index developed by J.P. Morgan, the Pensions Institute and Towers Watson,\textsuperscript{16} or the Xpect indices developed by Deutsche Boerse.\textsuperscript{17} The relative advantages and disadvantages of index-based versus indemnity-based swaps are discussed in Biffis and Blake (2010b). Assuming that the index admits the representation $I_t = \exp(-\int_0^t \mu_s^I ds)$, with $(\mu_t^I)_{t \geq 0}$ representing the intensity of mortality of a reference population, we can express the swap rate $\bar{p}^I$ by using the expression (2.3) again, but with the process $\mu$ replaced by $\mu^I$ and with $\tilde{p}_T$ replaced by the corresponding risk-adjusted survival probability $\tilde{p}_T^I$.

### 2.1 The marking-to-market (MTM) process

Longevity swaps are not currently exchange traded and there is no commonly accepted framework for counterparties to mark to market their positions.\textsuperscript{18} The presence of counterparty default risk and collateralization rules, however, makes the MTM procedure a very important feature of these transactions for at least three reasons. First, at each payment date, the difference between the variable and pre-set payment generates a cash inflow or outflow to the hedger, depending on the evolution of mortality. In the absence of basis risk (which is the case for indemnity-based solutions), these differences show a

\textsuperscript{16}See www.lifemetrics.com.

\textsuperscript{17}See www.xpect-index.com.

\textsuperscript{18}At the time of writing, LLMA was working on this issue.
pure ‘cashflow hedge’ of the longevity exposure in operation. Second, as market conditions change (e.g., mortality patterns, counterparty default risk), the MTM procedure could result in the swap switching status in the hedger’s balance sheet between that of an asset and that of a liability. This may have the implication that, even if the swap payments are expected to provide a good hedge against longevity risk, the hedger’s position may still turn into a liability if, for example, deterioration in the hedge supplier’s credit quality shrinks the expected present value of the variable payments. Third, for solvency requirements, it is important to value a longevity swap under extreme market/mortality scenarios (‘stress testing’). This means, for example, that even if a longevity swap qualifies as a liability on a market-consistent basis, it might still provide considerable capital relief when valued on a regulatory basis.

To illustrate some of these points, let us consider the hypothetical situation of an insurer A with a liability represented by a group of ten thousand 65-year-old annuitants drawn from the population of England & Wales in 1980. We assume that party A entered a 25-year pure longevity swap in 1980 and we follow the evolution of the contract until maturity. The population is assumed to evolve according to the death rates reported in the Human Mortality Database (HMD) for England & Wales.\(^{19}\) We assume that interest-rate risk is hedged away through interest rate swaps, locking in a rate of 5% throughout the life of the swap. The role of collateral is examined later on; here, we show how the hedging instrument operates from the point of view of the hedger. For this indemnity-based solution, the market value of each floating-for-fixed payment occurring at a generic date \(T\) can be computed by using the valuation formula

\[
S_t = n E_t^p \left\{ \exp \left( - \int_t^T \int_s^T \mu_d ds \right) \left( \frac{n - N_t}{n} \exp \left( - \int_t^T \mu_d ds \right) \right) \right\} - nB(t, T)p^N, \tag{2.5}
\]

for each time \(t\) in \([0, T]\) at which no default has yet occurred, with \(B(t, T)\) denoting the\(^{19}\)See www.mortality.org.
market value of a zero-coupon bond with time to maturity $T - t$, and $E_t^{\mathbb{F}}$ the conditional expectation given the information available at time $t$. As a simple benchmark case, we assume that market participants receive information from the HMD and use the Lee-Carter model to value longevity-linked cashflows. In other words, at each MTM date (including inception), longevity swap rates are based on Lee-Carter forecasts computed using the latest HMD information available (see Dowd et al., 2010a,b; Cairns et al., 2011, for a comprehensive analysis of alternative mortality models). Figure 1 illustrates the evolution of swap survival rates for an England & Wales cohort tracked from age 65 in 1980 to age 90 in 2005. It is clear that the systematic underestimation of mortality improvements by the Lee-Carter model in this particular example will mean that the hedger’s position will become increasingly in-the-money as the swap matures. This is shown in Figure 2. In practice the contract may allow the counterparty to cancel the swap or reset the fixed leg for a nonnegative fee, but we ignore these features in this example. Figure 2 also reports the sequence of cash inflows and outflows generated by the swap, which are lower ex-post than what was anticipated from an examination of the MTM basis. As interest rate risk is hedged — and again ignoring default risk for the moment — cash inflows/outflows arising in the backtesting exercise only reflect the difference between the realized survival rates and the swap rates locked in at inception. On the other hand, the swap’s market value reflects changes in market swap rates, which by assumption follow the updated Lee-Carter forecasts plotted in Figure 1 and differ from the realized survival rates. As is evident from the example depicted in Figure 2, the credit exposure of a longevity swap is close to zero at inception and at maturity, but may be sizable during the life of the swap, depending on the trade-off between changes in market/mortality conditions and the residual swap payments (amortization effect). The credit exposure is quantified by the replacement cost, i.e., the cost that the nondefaulting counterparty would have to incur at the default time to replace the instrument at market prices then available. As a simple example which predicts the
next section, let us introduce credit risk (but no default) and assume that in 1988 the credit spread of the hedge supplier widens across all maturities by 25 and 50 basis points. The impact of these two scenarios on the hedger’s balance sheet is dramatic, as shown again in Figure 2, demonstrating how MTM profits and losses can jeopardize even a well structured hedging position.

< Figure 1 about here >

< Figure 2 about here >

3 Counterparty default risk

The backtesting exercise of the previous section has demonstrated the importance of the hedge supplier’s credit risk and the marking to market procedure in assessing the value of a longevity swap to the hedger. As was emphasized in the introduction, however, a correct approach should allow for the fact that counterparty risk is bilateral. This is the case even when the hedger is a pension plan. Private sector defined benefit pension plans in countries such as the UK are founded on trust law and rely on a promise by (rather than a guarantee from) the sponsoring employer to pay the benefits to plan members. This promise is known as the ‘sponsor covenant’. The strength of the sponsor covenant depends on both the financial strength of the employer and the employer’s commitment to the scheme. To take into account the impact of the sponsor covenant, we use the sponsor’s default intensity, and refer to it as party A’s intensity of default. For large corporate pension plans, the intensity can be derived/extrapolated from spreads observed

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20In the UK, for example, The Actuarial Profession (2005, par. 3.2) defined the sponsor covenant as: “the combination of (a) the ability and (b) the willingness of the sponsor to pay (or the ability of the trustees to require the sponsor to pay) sufficient advance contributions to ensure that the scheme’s benefits can be paid as they fall due.” See also The Pensions Regulator (2009).
in corporate bond and CDS markets. For smaller plans, an analysis of the funding level and strategy of the scheme is required.

Assume that both party A (the hedger) and B (the hedge supplier) may default at random times $\tau^A, \tau^B$, admitting default intensities $(\lambda^A_t)_{t \geq 0}, (\lambda^B_t)_{t \geq 0}$. Defining by $\tau := \min(\tau^A, \tau^B)$, the default time of the swap transaction, we further assume that, on the event $\{\tau \leq T\}$, the nondefaulting counterparty, say party $i$, receives a fraction $\psi^j \in [0, 1]$ ($i \neq j$, with $i, j \in \{A, B\}$) of the market value of the swap before default, $S_{\tau^-}$, if she is in-the-money, otherwise she has to pay the full pre-default market value $S_{\tau^-}$ to the defaulting counterparty. We can then write the market value of a swap with notional amount $n$ as (e.g., Duffie and Huang, 1996)

$$S_0 = n E^P \left[ \exp \left( - \int_0^T (r_t + 1_{\{S_t < 0\}}(1 - \psi^A)\lambda^A_t + 1_{\{S_t \geq 0\}}(1 - \psi^B)\lambda^B_t)dt \right) (P - \mathbb{P}) \right],$$

(3.1)

where $P$ denotes the variable payment and the indicator function $1_H$ takes the value of unity if the event $H$ is true, zero otherwise. To understand the above formula, note that, in our setting, the risk-neutral valuation of a defaultable claim involves the use of a default-risk-adjusted short rate $r_t + \lambda^A_t + \lambda^B_t$ and dividend payment $\lambda^A_t(\psi^A 1_{S_t < 0} + 1_{S_t \geq 0}) + \lambda^B_t(\psi^B 1_{S_t \geq 0} + 1_{S_t < 0})$ determined by the recovery rules described above. As a result, the valuation formula (3.1) entails discounting at a spread above the risk-free given by

$$\Lambda_t := \lambda^A_t + \lambda^B_t - \lambda^A_t(\psi^A 1_{S_t < 0} + 1_{S_t \geq 0}) - \lambda^B_t(\psi^B 1_{S_t \geq 0} + 1_{S_t < 0})$$

$$= 1_{\{S_t < 0\}}(1 - \psi^A)\lambda^A_t + 1_{\{S_t \geq 0\}}(1 - \psi^B)\lambda^B_t,$$

showing a switching-type dependence on the characteristics of the counterparty that is out-of-the-money at each given time prior to default. The swap rate admits the
representation

\[ p^d = \mathbb{E}^{\mathbb{P}}[P] + \frac{\text{Cov}^{\mathbb{P}} \left( \exp \left( - \int_0^T (r_t + \Lambda_t) dt \right), P \right)}{\mathbb{E}^{\mathbb{P}} \left[ \exp \left( - \int_0^T (r_t + \Lambda_t) dt \right) \right]}, \]

(3.2)

and hence depends in a complex way not only on the interaction between the variable payments and risk factors such as interest rates, default intensities and recovery rates, but also on the path of the swap’s market value itself. When \( P \) does not include a demographic component, as in the case of interest rate swaps, the covariance term is typically negative. To see this, consider the case of the standard swap valuation formula obtained by assuming that both counterparties have the same default intensity (\( \lambda_t := \lambda^A_t = \lambda^B_t \)) and there is no recovery conditional on default (\( \psi^A = \psi^B = 0 \)). If the credit risk of the counterparties is equal to the average credit quality of the LIBOR panel, the discount rate in (3.2) is simply given by \( r + \lambda \), where \( \lambda \) is just the LIBOR-Treasury (TED) spread. For a swap paying the LIBOR rate, we would then have a negative covariance term and hence \( p^d < \mathbb{E}^{\mathbb{P}}[P] \). When \( P \) only includes a demographic component, as in expression (2.3) for example, we may expect the covariance term also to be negative, as longevity-linked payments are likely to be positively correlated with the credit quality of hedge suppliers\(^{21}\) and companies with significant pension liabilities.

The case of floating payments linked to both mortality and interest rates would then suggest a swap rate satisfying \( p^d < \mathbb{E}^{\mathbb{P}}[P] \). In the next section, we will show that this is not necessarily the case. To understand why, observe that, in the case of full recovery (\( \psi^A = \psi^B = 1 \)), expression (3.2) reduces to a default-free risk-neutral valuation formula, irrespective of the default intensities of the counterparties and of the costs involved by the credit enhancement tools needed to ensure that full recovery is indeed achieved upon default.

Counterparty risk can be mitigated through a number of credit enhancement tools, such as termination rights (e.g., credit puts and break clauses) or credit derivatives (e.g.,

\(^{21}\)This is a reasonable assumption for monoline insurers such as pension buyout firms, but might be less so for well diversified reinsurers.
CDSs and credit spread options). Here we focus on collateralization, a form of direct credit support requiring each party to post cash or securities when it is out of the money. For simplicity, we consider the case of cash, which is by far the most common type of collateral (e.g., ISDA, 2010a) and allows us to disregard close-out risk, the risk that the value of collateral may change at default. In the interest-rate swaps market, Johannes and Sundaresan (2007) find evidence of costly collateral by comparing swap market data with swap values based on portfolio of futures and forward contracts. We cannot carry out a similar exercise for longevity swaps, because there are no publicly available data for these transactions. On the other hand, we can quantify the funding gains/costs associated with the collateral flows originating from the marking to market procedure. We will therefore model explicitly the market value of a representative longevity-linked liability and use information on the funding cost of the counterparties to ‘synthesize’ the dynamics of collateral costs.

4 Collateralization

Collateral agreements reflect the amount of acceptable credit exposure that each party agrees to take on. We will consider simple collateral rules capturing the main features of the problem. Formally, let us introduce the pre-default collateral process\(^\text{22}\) \((C_t)_{t\geq 0}\), which indicates how much cash, \(C_t\), to post at each time \(t\) prior to default in response to changes in market conditions, including, in particular, the MTM value of the swap (we provide explicit examples below). Again, we develop our analysis from the point of view of the hedger, so that \(C_t > 0\) \((C_t < 0\) means that party A is holding (posting) collateral. Using the notation \(a^+ := \max(a, 0)\) and \(a^- := \max(-a, 0)\), the recovery rules take the following form:

- On the event \(\{\tau^A \leq \min(\tau^B, T)\}\) (hedger’s default), party B (the hedge supplier)

\(^{22}\)In other words, the actual collateral supporting the transaction is represented by \((1_{(t\geq 0)}C_t)_{t\geq 0}\), hence, we are not concerned with the value taken by \(C_t\) after default.
recovers any collateral received by the hedger an instant prior to default, $C_{τ_{A\rightarrow}}$, and pays the full MTM value of the swap to party A if $S_{τ_{A\rightarrow}} \geq 0$. The net flow to party A is then $S_{τ_{A\rightarrow}}^+ - C_{τ_{A\rightarrow}}^-$. 

- On the event $\{τ^B \leq \min(τ^A, T)\}$ (hedge supplier’s default), party A (the hedger) pays the full MTM value of the swap to party B if $S_{τ_{B\rightarrow}}^- < 0$, and recovers any collateral received by B an instant prior to default, $C_{τ_{B\rightarrow}}^+$. The net flow to party A can then be written as $-S_{τ_{B\rightarrow}}^- + C_{τ_{B\rightarrow}}^+$. 

- Whenever the nondefaulting counterparty, say $A$, is out-of-the-money, payment of the full MTM value of the swap is accomplished by party A recovering the extra amount $(S_{τ_{\rightarrow}} - C_{τ_{\rightarrow}})^+$ in case of overcollateralization, or by party A paying the extra amount $(C_{τ_{\rightarrow}} - S_{τ_{\rightarrow}})^+$ in case of undercollateralization. In case of full collateralization, party A simply loses any collateral posted with B.

To obtain neater results, it is convenient to express the collateral before default of either party as a fraction of the MTM value of the swap,

$$C_t = (c_t^B 1_{\{S_t\geq 0\}} + c_t^A 1_{\{S_t<0\}}) S_t^-,$$  \hspace{1cm} (4.1)

where $c^A, c^B$ are two nonnegative left-continuous processes giving the fraction of the MTM value of the swap that is posted as collateral by party A or B, respectively. Finally, we introduce a nonnegative continuous process $(δ_t)_{t \geq 0}$ representing the yield on collateral, in the sense that holding/posting collateral of amount $C_t$ yields/costs instantaneously the net amount $δ_t C_t$ (after rebate). Instead of capturing simultaneously the perspective of both counterparties with $δ$, it may be convenient to introduce some asymmetry by considering $δ_t = δ_t^A 1_{\{S_t<0\}} + δ_t^B 1_{\{S_t\geq 0\}}$, so that $δ_t^A (δ_t^B)$ can be interpreted as the opportunity cost of posting collateral for party A (B) when it is out-of-the-money. Denoting by $\overline{p}^t$ the swap rate available in case of collateralization, we can write the MTM value of the swap as in (3.1), but with the spread $Δ$ now replaced by $\overline{p}^t$.  

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for a proof)

\[ \Gamma_t = \lambda_t^A (1 - c_t^A) 1_{\{S_t < 0\}} + \lambda_t^B (1 - c_t^B) 1_{\{S_t \geq 0\}} - (\delta_t^A c_t^A 1_{\{S_t < 0\}} + \delta_t^B c_t^B 1_{\{S_t \geq 0\}}) \]  

(4.2)

In the above expression, we recognize the typical features of valuation formulae for credit-risky securities (e.g., Duffie and Huang, 1996; Bielecki and Rutkowski, 2002): the first two terms account for the fractional recovery of the swap MTM value in case of default of the counterparty, the third one for the costs incurred when posting collateral before default. The swap rate \( \mathbf{p}^c \) can finally be written as in (3.2) by using the discount rate \( r + \Gamma \). We now examine simple special cases to understand better the role of collateral in shaping swap rates.

### 4.1 Full collateralization

Consider the collateral rule obtained by setting \( c^A = c^B = 1 \) and \( \delta^A = \delta^B = \delta \), meaning that the full MTM value of the swap is received/posted as collateral depending on whether the marking-to-market process results in a positive/negative value for \( S_t \). As we consider cash collateral, default is immaterial. In contrast with section 3, however, the expression for the swap MTM value does not reduce to the usual default-free, risk-neutral valuation formula in general, unless collateral costs are zero:

\[ \mathbf{p}^c = E^{\mathbb{P}} \left[ P \right] + \frac{Cov^{\mathbb{P}} \left( \exp \left( \int_0^T (\delta_t - r_t) dt \right), P \right)}{E^{\mathbb{P}} \left[ \exp \left( \int_0^T (\delta_t - r_t) dt \right) \right]}. \]  

(4.3)

If the cost of collateral is positively dependent on interest rates, we expect the swap rate to be higher than \( \mathbf{p}^d \) in expression (3.2), even if the floating payment is only linked to interest rates, reflecting the fact that (costly) collateralization commands a premium for the payer of the fixed rate (see Johannes and Sundaresan, 2007). The intuition is that the party paying the floating rate will have to both post collateral and incur higher
funding costs when the floating rate increases. As was emphasized in the introduction, in longevity space, the cost of collateral is positively dependent on mortality improvements, but longevity-linked liabilities are more capital intensive in low interest rate environments (due to lower discounting of future cashflows). The combined impact of these two effects is difficult to quantify, and we may have situations in which \( \bar{p} \geq E[\bar{P}] \) even if \( E[\bar{P}] \geq p \).

4.2 Partial collateralization

According to ISDA (2010a), it is typical for collateral agreements to specify collateral triggers based on the market value of the swap or other relevant variables (credit ratings, credit spreads, etc.) crossing pre-specified threshold levels. In indemnity-based longevity swaps, however, it is more common to define collateral thresholds in terms of mortality forecasts based on a model agreed at contract inception and to monitor the deaths in the hedger’s population than in terms of the market value of the swap. This is due to both the re-estimation risk affecting any given mortality model and the presence of substantial model risk, which most likely would prevent the counterparties from agreeing on a common model at future dates. The following are relevant (if somewhat stylized) examples that are useful for our case:

a) Consider the collateral rule obtained by setting \( c_t^B = 1\{S_t - \bar{s}(t) \geq \bar{s}(t)\} \) and \( c_t^A = 1\{S_t - \bar{s}(t) \leq s(t)\} \) for continuous functions \( \bar{s}, \bar{\bar{s}} \) defined on \([0, T]\) and satisfying \( \bar{s} \leq \bar{\bar{s}} \), meaning that the hedge supplier (hedger) is required to post full collateral if the swap’s MTM value is above (below) the appropriate time-dependent threshold. More general collateral rules can be obtained by setting \( c_t^B = \gamma_t^B 1\{S_t - \bar{s}(t) \geq \bar{s}(t)\} \) and \( c_t^A = \gamma_t^A 1\{S_t - \bar{s}(t) \leq s(t)\} \), for suitable processes \( \gamma^A, \gamma^B \) depending on prevailing market conditions, such as the credit standing of the counterparties.

b) Setting \( c_t^B = 1\{N_t - \alpha(t) \leq \beta(t)\} \) and \( c_t^A = 1\{N_t - \alpha(t) \geq \beta(t)\} \), for continuous functions \( \alpha \) and \( \beta \).
satisfying $0 \leq \alpha \leq \beta \leq n$, we see that the hedge supplier (hedger) is required to post full collateral if realized deaths are below (above) the relevant threshold. Given the above considerations, this collateralization rule is more suitable for an indemnity-based longevity swap.

c) For an index-based swap, it may be more convenient to work with the mortality intensity $\mu^{I}$ of the reference population (see section 2) and set $c_{t}^{B} = 1_{\{\int_{0}^{t} \mu^{I} ds \leq \alpha(t)\}}$ and $c_{t}^{A} = 1_{\{\int_{0}^{t} \mu^{I} ds \geq \beta(t)\}}$ for continuous functions $a, b$ satisfying $0 \leq a \leq b$. This means that collateral posting is triggered at each time $t$ if the realized value of the longevity index, $\exp\left(-\int_{0}^{t} \mu^{I} ds\right)$, falls outside the interval $(\exp(-b(t)), \exp(-a(t)))$.

d) As was emphasized in section 2.1, the severity of counterparty risk depends on the credit quality of the counterparties. This is why collateralization agreements may set collateral thresholds that explicitly depend on credit ratings or CDS spreads. A simple example of this practice can be obtained as a special case of formulation (a) by setting $c_{t}^{B} = 1_{\{N_{t^-} \leq \alpha(t)\} \cup \{\lambda_{t}^{B} \geq \lambda\}}$, $c_{t}^{A} = 1_{\{N_{t^-} \geq \beta(t)\} \cup \{\lambda_{t}^{A} \geq \lambda\}}$, meaning that, at each time $t$, the hedger (hedge supplier) receives collateral when either realized deaths fall below the level $\alpha(t)$ ($\beta(t)$) or the hedge supplier’s (hedger’s) default intensity overshoots a given threshold $\lambda \geq 0$. Note that both $c^{A}$ and $c^{B}$ can be non zero at the same time (for example on the event $\{N_{t^-} \leq \alpha(t)\} \cap \{\lambda_{t}^{A} \geq \lambda\}$), but expression (4.1) ensures that only the party out-of-the-money will have to post collateral.

4.3 Computing the swap rate

The recursive nature of swap valuation formulae in the case of bilateral and asymmetric counterparty risk was already noted by Duffie and Huang (1996). By modeling the recovery rates and the difference in counterparties’ credit spreads in reduced form, however, they could use a simpler iterative procedure to determine swap rates. Here, we explic-
itly allow for the impact of collateral and the marking-to-market process in the pricing functional, and hence need a different approach. Working in a Markov setting, we let $X$ denote the state variable process and adopt a Least-Squares Monte Carlo approach. Exploiting the structure of the mortality model outlined in section 2 and in the appendix, we do not model death/default times explicitly but just rely on the mortality/default intensities (see algorithm 2 in Bacinello et al., 2010, for example). The procedure involves the following steps:

**Step 1.** For fixed swap rate $p_{c_i}^*$, generate $M$ simulated paths of $X$ under $\tilde{P}$ along the time grid $\mathcal{T} := \{0 < t_1, t_2, \ldots, t_n = T\}$. Denote by $S_{t_j}^{m,i}$ the MTM value of the swap and by $f_{t_j}^{i,m}$ the cashflows originating from the swap (collateral flows and swap payments) at time $t_j$ on path $m$ and for given swap rate $p_{c_i}^*$.

**Step 2.** Compute recursively the value of the swap at time $t_j$ (for $j = n-1, \ldots, 0$ with $t_0 = 0$) as $S_{t_j}^{m,i} = \beta_j^* \cdot e(X_{t_j}^{m})$, where $e(x) := (e_1(x), \ldots, e_H(x))^T$ and $\{e_1, \ldots, e_H\}$ is a finite set of functions taken from a suitable basis, and $\beta_j^*$ is given by

$$\beta_j^* = \arg \min_{\beta_j \in \mathbb{R}^H} \sum_{m=1}^{M} \left( S_{t_{j+1}}^{i,m} + f_{t_{j+1}}^{i,m} - \beta_j \cdot e(X_{t_j}^{m}) \right).$$

At each time $t_j$, use $S_{t_j}^{m,i}$ to check whether the collateral thresholds are triggered and determine the corresponding amount of collateral and associated costs.

**Step 3.** Iterate the above procedure over different values for $p_{c_i}^*$ until a candidate swap rate $p_{c_i}^*$ is found, such that the initial price of the swap, $\frac{1}{M} \sum_{m=1}^{M} S_{t_0}^{m,i}^*$, is close enough to zero. Set $p^* = p_{c_i}^*$.

## 5 Examples

We use a continuous-time model for the risk-free yield curve, the LIBOR and mortality rates, as well as for the cost of collateral. The credit risk of party B (the hedge supplier)
is assumed to be equal to the average credit quality of the LIBOR panel, so that the TED spread could be seen as its default intensity if there were zero recovery upon default (see section 3). We then set $\lambda^A = \lambda^B + \Delta$ and consider two cases: party A is either of the same credit quality as party B ($\Delta = 0$) or is more credit-risky ($\Delta > 0$).

We consider a Markov setting, and describe the evolution of uncertainty by a six-dimensional state variable vector $X$ with the Gaussian dynamics reported in appendix B. The first four components describe: the short rate, $r = X^{(1)}$, assumed to revert to the long-run central tendency factor $X^{(2)}$, representing the slope of the risk-free yield curve; the TED spread $X^{(3)}$, so that the LIBOR rate is given by $X^{(1)} + X^{(3)}$; and the net yield on collateral in the interest-rate swap market, $X^{(4)}$. The remaining two components describe the log-intensity of mortality of a given population, $\log \mu = X^{(6)}$, and the yield on collateral attached to longevity risk business, $X^{(5)}$. Under the assumption of independence between the interest rate market and mortality, we can estimate separately the dynamics of the two groups of factors ($X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}$) and $X^{(6)}$. For the first group, we use the estimates of Johannes and Sundaresan (2007) who use weekly Treasury and swap data from 1990 to 2002 to obtain the parameter values reported in table 2. For the intensity $\exp(X^{(6)}_t)$, we use a continuous-time version of the Lee-Carter model; see the appendix for details. As we do not have any publicly available transaction data from the longevity swap market to proxy $X^{(5)}$, we use information on credit markets (funding costs) and regulatory requirements (capital charges). In particular, as a first example, we simply take $\delta^B = X^{(3)}$ and $\delta^A = X^{(3)} + \Delta$, meaning that the net collateral costs coincide with each party’s borrowing rate net of the risk-free rate (assuming it is rebated). In the case of asymmetric default risk, we take $\Delta = 0.01$. In a second example, discussed in detail below, we simulate the capital charges arising from holding a representative longevity-linked liability to estimate the dynamics of $X^{(5)}$. In both cases, we compute the longevity swap rates for a 25-year swap written on a population of 10,000 US males aged 65 at the beginning of 2008.
In figure 3, we plot the swap curves obtained for different collateralization rules against the percentiles of survival rate improvements based on Lee-Carter forecasts. We see that margins are positive and increasing with payment maturity in the case of symmetric default risk, for both uncollateralized and fully collateralized transactions. As soon as we introduce asymmetry in default risk ($\Delta > 0$), however, margins widen in the case of no collateralization, reflecting the fact that the hedger needs to pay an additional premium on account of its higher credit risk. In the case of full collateralization, the hedge supplier benefits from the negative dependence between funding costs and collateral amounts discussed before: equilibrium swap rates are pushed lower and produce a negative margin on best estimate swap rates. In figure 4, we examine the swap margins induced by one-sided collateralization in the case of asymmetric default risk. When only the hedge supplier has to post full collateral, swap rates are higher than best estimate survival probabilities, meaning that the hedger has to compensate the counterparty for bearing the cost of risk mitigation on its side, while remaining exposed to the hedger’s default risk. The opposite is true when it is the hedger who has to post full collateral when out-of-the-money. In this case, swap margins are clearly negative, and again increasing in payment maturity.

Plotting the swap rate margins against best estimate mortality improvements allows one to interpret the swap rates as outputs of a pricing functional based on adjustments to a reference mortality model. On the other hand, it is useful to look at the swap spreads (in basis points) to have an idea of how they compare with those emerging in other transactions. In table 3, we report the spreads relative to some key maturities and collateralization rules: in absolute value, the spreads never exceed those documented in the interest-rate swap market for fully collateralized transactions with maturities of at most 10 years. For example, in the case of full collateralization on the hedge supplier’s
side only, longevity swap rates for 15- to 25-year maturities command a premium comparable with that found for bilaterally collateralized interest-rate swaps with maturities of 6.5 to 10 years (see Johannes and Sundaresan, 2007). These observations suggest that the impact of collateralization on longevity swap rates is comparable with, and possibly even milder than, that found for the far more liquid interest-rate swap market. There are two main reasons for this result. First, in contrast with markets where rehypothecation of collateral is the norm, regulation makes collateral less fungible in longevity swap transactions (through segregation, for example). Second, as discussed above, interest rate risk and longevity risk impact swap margins in opposite directions, thus diluting the impact of collateralization on longevity swap rates.

< Table 3 about here >

< Figure 3 about here >

< Figure 4 about here >

In a second example, we ‘synthesize’ the dynamics of $X^{(5)}$ by using information on regulatory requirements to quantify the capital charges accruing to the counterparties during the life of the swap. In particular, we use the following bottom-up procedure:

Step 1: We simulate several paths of the factors $X^{(1)}, \ldots, X^{(4)}$ and $X^{(6)}$ along a time grid $\hat{T} := \{t_1, t_2, \ldots, t_k\}$ (with $t_1 > 0$ and $t_k = \hat{T} > t_1$) and under the pricing measure $\tilde{\mathbb{P}}$. Consistent with focusing on the baseline case of $\tilde{p}_T = p_T$ discussed in section 2, the $\tilde{\mathbb{P}}$-dynamics of $X^{(6)}$ is assumed to be the same as under the physical measure.

Step 2: The paths simulated in the previous step are used to compute, at each date $t \in \hat{T}$, the regulatory capital needed by an insurer to hold the liability $n - N_{t+T}$, where $\overline{T} < \hat{T}$ is a representative maturity proxying the average duration of longevity-linked
liabilities in the longevity swap market. We use $T = 15$ (years) for our example. To compute the capital requirement, we use the Solvency II framework, which is based on the 99.5% value-at-risk of the net assets over a one-year horizon. For simplicity, we assume holders of longevity exposures to be invested in cash. The distribution of the one-year-ahead market-consistent value of the liability usually requires nested simulation, unless a simplified approach is adopted. In our setting, market-consistent discount factors can be computed analytically based on the one-year-ahead simulated realizations. We use Least-Squares Monte Carlo (see section 4.3) for the expected number of survivors.\(^{23}\)

*Step 3:* We use the simulated capital charges obtained in the previous step to compute the gains/costs incurred to reduce/increase capital at each time step along each simulated path. We assume that capital charges are funded at the counterparties’ funding cost, plus a spread of 6% to reflect that opportunity cost of diverting to an individual liability funds that could be used to support insurance business at aggregate level. The simulated realizations of the opportunity cost of capital are used to estimate the dynamics of $X^{(5)}$ reported in the appendix. The parameter estimates are included in table 2.

In the case of symmetric collateralization, we find results comparable with those obtained by using the counterparties’ funding costs for the process $\delta$. However, figure 5 shows that margins increase (decrease) considerably when one-sided collateralization on the hedge supplier’s (hedger’s) side is considered. The reason is that the party required to post collateral explicitly takes into account tail events in computing collateral costs, whereas in figure 4 funding costs where computed on the basis of the market value of the longevity swaps.

Finally, we study the sensitivity of the longevity swap spreads to the volatility of the net collateral cost $X^{(5)}$. To close off the interest-rate risk channel, we fix the factors $X^{(1)}, X^{(2)}$ equal to their long-run means. Table 4 reports the results obtained for different values of the volatility parameter $\sigma_5$ in the case of symmetric default risk and bilateral

\(^{23}\)See Stevens, De Waegenaere and Melenberg (2010a) for other approximation methods in the context of Lee-Carter forecasts.
full collateralization. We see that spreads increase dramatically for large values of the volatility parameter, but are comparable with those found in the interest-rate swap market for reasonable volatility levels (e.g., below 5%; see Johannes and Sundaresan, 2007).

Table 4 about here

Figure 5 about here

6 Conclusion

In this study, we have provided a framework for understanding and quantifying the cost of bilateral default risk and collateral strategies on longevity risk solutions. The results address the concerns aired by potential hedgers regarding how to measure the trade-off between the hedge effectiveness of longevity-linked instruments and the counterparty risk they involve. We have described a methodology for pricing longevity swaps that explicitly takes into account the dynamics of the marking-to-market process, the collateral flows it generates, and the costs associated with the posting of collateral. We have shown how collateral strategies can mitigate if not eliminate counterparty risk, but inevitably introduce an extra cost that must be borne by the hedge supplier or by the hedger, depending on whether it is longevity risk or interest rate risk that has a stronger impact on the opportunity cost of collateral. Our most significant and useful finding is that the overall cost of the collateralization strategies in the longevity swap market is comparable with, and often much smaller than, that found in the (currently) much more liquid interest-rate swap market. Hence, there is no reason to suppose that counterparty risk in the longevity swap market will provide an insurmountable barrier to the further development of this market. Our analysis accordingly provides a robust framework for
comparing the costs of credit enhancement in indemnity-based longevity swaps with the
benefits offered by competing solutions such as securitization and indexed swaps.

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A Details on the setup

We take as given a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\), and model the death times in a population of \(n\) individuals (annuitants or pensioners) as stopping times \(\tau_1, \ldots, \tau_n\). This means that at each time \(t\) the information carried by \(\mathcal{F}_t\) allows us to state whether each individual has died or not. The hedger’s liability is given by the random variable \(\sum_{i=1}^{n} 1_{\{\tau_i > T\}}\), which can be equivalently written as \(n - \sum_{i=1}^{n} 1_{\{\tau_i \leq T\}} = n - N_T\) (recall that the indicator function \(1_H\) takes the value of unity if the event \(H\) is true, zero otherwise). We assume that death times coincide with the first jumps of \(n\) conditionally Poisson processes with common random intensity of mortality \((\mu_t)_{t \geq 0}\) under both \(\mathbb{P}\) and an equivalent martingale measure \(\tilde{\mathbb{P}}\) (see Biffis et al., 2010, for details). The expected number of survivors over \([0,T]\) under the two measures can then be expressed as \(E^{\mathbb{P}}[\sum_{i=1}^{n} 1_{\{\tau_i > T\}}] = np_T\) and \(E^{\tilde{\mathbb{P}}}[\sum_{i=1}^{n} 1_{\{\tau_i > T\}}] = n\tilde{p}_T\), with \(p_T\) and \(\tilde{p}_T\) given by the expectation (2.1) computed under the relevant probability measure.

Consider any stopping time \(\tau_i\) satisfying the above assumptions, an integrable ran-
dom variable $Y \in \mathcal{F}_T$ and a bounded process $(X_t)_{t \in [0,T]}$ such that each $X_t$ is measurable with respect to $\mathcal{F}_{t-}$, the information available up to, but not including, time $t$. Then a security paying $Y$ at time $T$ in case $\tau^i > T$ and $X_{\tau^i}$ at time $\tau^i$ in case $\tau^i \leq T$ has time-zero price

$$E^{\tilde{P}} \left[ \int_0^T \exp \left( - \int_0^s (r_t + \mu_t) dt \right) X_s \mu_s ds + \exp \left( - \int_0^T (r_t + \mu_t) dt \right) Y \right].$$

Consider now two stopping times $\tau^i, \tau^j$, with intensities $\mu^i, \mu^j$, jointly satisfying the above assumptions (i.e., they are the first jump times of the components of a bivariate conditionally Poisson process). A security paying $Y$ at time $T$ in case neither stopping time has occurred (i.e., $\min(\tau^i, \tau^j) > T$) and $X_{\tau^i}$ at time $\tau^i$ in case the first occurrence is at time $t \in [0,T]$ (i.e., $t = \min(\tau^i, \tau^j)$) has time-zero price given by the same formula, with $\mu_t$ replaced by $\mu^i_t + \mu^j_t$. This follows from the fact that the stopping time $\min(\tau^i, \tau^j)$ is the first jump time of a conditionally Poisson process with intensity $(\mu^i_t + \mu^j_t)_{t \geq 0}$ (e.g., Bielecki and Rutkowski, 2002). The expressions presented in sections 2-4 all follow from these simple results.

**Proof of expression (4.2).** Let $(\delta^A_t, \delta^B_t)_{t \geq 0}$ denote the opportunity costs of collateral for the two parties, meaning that holding collateral of amount $C_t$ provides an instantaneous yield equal to $\delta^B_t C_t^+ - \delta^A_t C_t^-$ (we use the notation $a^+ := \max(a, 0)$, $a^- := -\min(a, 0)$).

We assume that collateral is bounded and $C_t$ is measurable with respect to $\mathcal{F}_{t-}$ for all $t \in [0,T]$. Parties A and B are assumed to have death (default) times satisfying the properties reviewed above, in particular having intensities $\lambda^A, \lambda^B$. Recalling the recovery rules described in section 4, we can then write:

$$S_0 = E^{\tilde{P}} \left[ \exp \left( - \int_0^T (r_t + \lambda^A_t + \lambda^B_t) dt \right) (P - p^d) \right]$$

$$+ E^{\tilde{P}} \left[ \int_0^T \exp \left( - \int_0^s (r_t + \lambda^A_t + \lambda^B_t) dt \right) (\lambda^A_s (S^+_s - C^-_s) + \lambda^B_s (C^+_s - S^-_s)) ds \right]$$

$$+ E^{\tilde{P}} \left[ \int_0^T \exp \left( - \int_0^s (r_t + \lambda^A_t + \lambda^B_t) dt \right) (\delta^B_s C^+_s - \delta^A_s C^-_s) ds \right].$$
Using representation (4.1), the amount recovered by the nondefaulting counterparty at time \( \tau = \min(\tau^A, \tau^B) \leq T \) is

\[
1_{\{\tau = \tau^A\}} S_{\tau^A} (c^A \tau^A 1_{\{S_{\tau^A} < 0\}} + 1_{\{S_{\tau^A} \geq 0\}}) + 1_{\{\tau = \tau^B\}} S_{\tau^B} (c^B \tau^B 1_{\{S_{\tau^B} \geq 0\}} + 1_{\{S_{\tau^B} < 0\}}),
\]

where we see that \( c^A, c^B \) replace the recovery rates \( \psi^A, \psi^B \) introduced in section 3. We can then write

\[
S_0 = E^\mathbb{P} \left[ \exp \left( - \int_0^T (r_t + \lambda^A_t + \lambda^B_t) dt \right) \left( P - p_d \right) \right] + E^\mathbb{P} \left[ \int_0^T \exp \left( - \int_0^s (r_t + \lambda^A_t + \lambda^B_t) dt \right) (\lambda^A_s + (\lambda^B_s + \delta^B_s) c^B_s) S^+_s - (\lambda^B_s + (\lambda^A_s + \delta^A_s) c^A_s) S^-_s \right] ds
\]

\[
= E^\mathbb{P} \left[ \exp \left( - \int_0^T (r_t + \Gamma_t) dt \right) \left( P - p_d \right) \right],
\]

which is nothing other than the usual risk-neutral valuation formula for a security with terminal payoff \( S_T = P - p_d \) paying continuously a dividend equal to a fraction

\[
(\lambda^A_s + (\lambda^B_s + \delta^B_s) c^B_s) 1_{\{S_t \geq 0\}} + (\lambda^B_s + (\lambda^A_s + \delta^A_s) c^A_s) 1_{\{S_t < 0\}}
\]

of the security’s market value an instant before each \( t \in [0, T] \). Subtracting the dividend rate from \( \lambda^A + \lambda^B \) and rearranging terms we obtain expression (4.2) for \( \Gamma \).
B Details on the numerical examples

The numerical examples are based on a six-dimensional state variable process \( X = (X^{(1)}, \ldots, X^{(6)})^T \) having \( \tilde{\mathbb{P}} \)-dynamics

\[
\begin{align*}
\mathrm{d}X^{(1)}_t &= \left( k_1(X^{(2)}_t - X^{(1)}_t) - \eta^1 \right) \mathrm{d}t + \sigma_1 \mathrm{d}W^1_t \\
\mathrm{d}X^{(2)}_t &= \left( k_2(\theta_2 - X^{(2)}_t) - \eta^2 \right) \mathrm{d}t + \sigma_2 \mathrm{d}W^2_t \\
\mathrm{d}X^{(3)}_t &= \left( \kappa_3(\theta_3 - X^{(3)}_t) + \kappa_{3,1}(X^{(1)}_t - \theta_2) + \kappa_{3,4}(X^{(4)}_t - \theta_4) - \eta_3 \right) \mathrm{d}t + \sigma_3 \mathrm{d}W^3_t \\
\mathrm{d}X^{(4)}_t &= \left( \kappa_4(\theta_4 - X^{(4)}_t) + \kappa_{4,1}(X^{(1)}_t - \theta_2) + \kappa_{4,2}(X^{(2)}_t - \theta_2) - \eta_4 \right) \mathrm{d}t + \sigma_4 \mathrm{d}W^4_t \\
\mathrm{d}X^{(5)}_t &= \left( \kappa_5(\theta_5 - X^{(5)}_t) + \kappa_{5,1}(X^{(1)}_t - \theta_2) + \kappa_{5,2}(X^{(2)}_t - \theta_2) + \kappa_{5,3}(X^{(3)}_t - \theta_3) \\
&\quad + \kappa_{5,4}(X^{(4)}_t - \theta_4) + \kappa_{5,6}(X^{(6)}_t - E_0[X^{(6)}_t]) - \eta_5 \right) \mathrm{d}t + \sigma_5 \mathrm{d}W^5_t \\
\mathrm{d}X^{(6)}_t &= \left( A(t) + B(t)(X^{(6)}_t - a(t)) \right) \mathrm{d}t + \sigma_6(t) \mathrm{d}W^6_t,
\end{align*}
\]

where \( W = (W^1, \ldots, W^6)^T \) is a standard \( \tilde{\mathbb{P}} \)-Brownian motion, the constants \( \eta^i \) represent market prices of risk and the functions \( A(\cdot), B(\cdot), \sigma_6(\cdot) \) are defined below. The \( \mathbb{P} \)-dynamics are obtained by removing the market prices of risk from the drifts of the relevant factors and replacing the innovations with the corresponding \( \mathbb{P} \)-Brownian innovations. We assume that \( X^{(6)} \) has the same dynamics under the physical and the pricing probability measures, consistent with our baseline case of a swap rate equal to \( p_T \) for each \( T \) in the absence of collateral. The Brownian innovations are uncorrelated, with the exception of the pair \( W^1, W^2 \), whose instantaneous correlation is denoted by \( \rho_{1,2} \).

For the first four factors, we use data from Johannes and Sundaresan (2007) who rely on a two-stage maximum likelihood procedure based on weekly data sampled on Wednesdays, from 1990 to 2002, and set the long-run mean of \( X^{(3)} \) equal to the average of the 3-month TED spread over the sampling period. For the log-intensity \( X^{(6)} \), we use the mortality model described below, and assume that the Brownian component \( W^6 \) is uncorrelated with the other ones. The intensity of mortality is modeled
using a continuous-time version of the Lee-Carter model (see Biffis et al., 2010). We first use the annual central death rates \( \{m_{y,s}\} \) for US and UK males from the Human Mortality Database to estimate the model \( m_{y,s} = \exp(\alpha(y) + \beta(y)K_s) \) for dates \( s = 1961, 1962, \ldots, 2007 \) and ages \( y = 20, 21, \ldots, 89 \) with Singular Value Decomposition. The resulting estimates for \( K \) are then fitted with the process \( K_{s+1} = \delta K_s + \sigma K \varepsilon \), with \( \varepsilon \sim N(0,1) \). For fixed age \( x = 65 \), the estimates for \( \{\hat{\alpha}(x+h), \hat{\beta}(x+h)\}_{h=0,1,...} \) are interpolated with differentiable functions \( a(t), b(t) \). The functions \( A, B, \sigma_6 \) are finally obtained by setting \( A(t) = a'(t) + b(t)\delta_K \), \( B(t) = b'(t)b(t)^{-1} \) and \( \sigma_6(t) = b(t)\sigma_K \). The expectation appearing in the drift of \( X^{(5)} \) ensures that the longevity capital charges react to departures of realized mortality from the term structure of survival rates estimated at inception.

To estimate the dynamics of \( X^{(5)} \), the component of collateral costs related to longevity risk, we implement the procedure discussed in section 5, setting the duration \( T \) of the representative liability equal to 15. We simulate forward all of the other state variables, and at each time step we compute the opportunity cost of capital arising from the capital charges accruing to the hedge supplier based on the simulated mortality and market conditions. We assume that funding occurs at the LIBOR rate plus a fixed spread of 6%, a reasonable value for the cost of internal capital. To obtain the net cost of collateral, we take into account the rebate of the risk-free rate. We estimate the dynamics of \( X^{(5)} \) on each simulated path. We set the parameter \( \theta_5 \) equal to the average of \( X^{(5)} \) along the simulated path. The parameter estimates are computed for each simulated path and then averaged across all simulations. The estimates are reported in table 2.
## Tables and figures

<table>
<thead>
<tr>
<th>Date</th>
<th>Hedger</th>
<th>Size</th>
<th>Term (yrs)</th>
<th>Type</th>
<th>Interm./supplier</th>
</tr>
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<tr>
<td>Jan 08</td>
<td>Lucida</td>
<td>Not disclosed</td>
<td>10</td>
<td>indexed</td>
<td>JP Morgan</td>
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<td>40</td>
<td>indemnity</td>
<td>JP Morgan</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>ILS funds</td>
</tr>
<tr>
<td>Feb 2009</td>
<td>Abbey Life</td>
<td>GBP 1.5bn</td>
<td>run-off</td>
<td>indemnity</td>
<td>Deutsche Bank</td>
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<tr>
<td>Mar 2009</td>
<td>Aviva</td>
<td>GBP 475m</td>
<td>10</td>
<td>indemnity</td>
<td>Royal Bank of Scotland</td>
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<tr>
<td>Jun 2009</td>
<td>Babcock</td>
<td>GBP 750m</td>
<td>50</td>
<td>indemnity</td>
<td>Credit Suisse</td>
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<tr>
<td>Jul 2009</td>
<td>RSA</td>
<td>GBP 1.9bn</td>
<td>run-off</td>
<td>indemnity</td>
<td>Pacific Life Re</td>
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<tr>
<td>Dec 2009</td>
<td>Berkshire Council</td>
<td>GBP 750m</td>
<td>run-off</td>
<td>indemnity</td>
<td>Goldman Sachs (Rothesay Life)</td>
</tr>
<tr>
<td>Feb 2010</td>
<td>BMW UK</td>
<td>GBP 3bn</td>
<td>run-off</td>
<td>indemnity</td>
<td>Deutsche Bank Paternoster</td>
</tr>
<tr>
<td>Dec 2010</td>
<td>Swiss Re (Kortis bond)</td>
<td>USD 50m</td>
<td>8</td>
<td>indexed</td>
<td>ILS funds</td>
</tr>
<tr>
<td>Feb 2011</td>
<td>Pall (UK) Pension Fund</td>
<td>GBP 70m</td>
<td>10</td>
<td>indexed</td>
<td>JP Morgan</td>
</tr>
</tbody>
</table>

Table 1: Publicly announced longevity swap transactions 2008-2011.
Table 2: Parameter values for the dynamics of $X$ given in Appendix B. The estimates for $X^{(5)}$ are based on the assumption that capital increases are funded by counterparties at 6% plus the LIBOR rate.

<table>
<thead>
<tr>
<th>$\kappa_1$</th>
<th>$\eta_1$</th>
<th>$\sigma_1$</th>
<th>$\delta_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.969</td>
<td>-0.053</td>
<td>0.008</td>
<td>-0.888</td>
</tr>
<tr>
<td>$\kappa_2$</td>
<td>$\eta_2$</td>
<td>$\sigma_2$</td>
<td>$\delta_K$</td>
</tr>
<tr>
<td>0.832</td>
<td>-0.014</td>
<td>0.155</td>
<td>-0.761</td>
</tr>
<tr>
<td>$\kappa_3$</td>
<td>$\eta_3$</td>
<td>$\sigma_3$</td>
<td>$\sigma_K$</td>
</tr>
<tr>
<td>1.669</td>
<td>0.007</td>
<td>0.009</td>
<td>1.156</td>
</tr>
<tr>
<td>$\kappa_4$</td>
<td>$\eta_4$</td>
<td>$\sigma_4$</td>
<td>$\sigma_K$</td>
</tr>
<tr>
<td>0.045</td>
<td>0.055</td>
<td>0.010</td>
<td>1.078</td>
</tr>
<tr>
<td>$\kappa_5$</td>
<td>$\kappa_{5,1}$</td>
<td>$\sigma_5$</td>
<td>$\delta_K$</td>
</tr>
<tr>
<td>0.990</td>
<td>0.147</td>
<td>0.690</td>
<td>-0.761</td>
</tr>
</tbody>
</table>

Table 3: Second example in section 5: swap spreads $p_T - p_T^{*}$ (in basis points) for different collateralization rules and maturities. The LSMC procedure uses 5000 paths over a quarterly grid with polynomial basis functions of order 3, and is repeated for 100 seeds.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>$c^A = 0$</th>
<th>$c^B = 0$</th>
<th>$c^A = 1$</th>
<th>$c^B = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 yrs</td>
<td>0.03</td>
<td>11.34</td>
<td>-11.76</td>
<td>0.05</td>
</tr>
<tr>
<td>20 yrs</td>
<td>1.11</td>
<td>19.93</td>
<td>-17.94</td>
<td>0.86</td>
</tr>
<tr>
<td>25 yrs</td>
<td>1.50</td>
<td>21.25</td>
<td>-18.35</td>
<td>1.24</td>
</tr>
</tbody>
</table>

Table 4: Sensitivity with respect to parameter $\sigma_5$: we compute 25-year swap rates and spreads (in basis points) under full collateralization by setting $X^{(1)}, X^{(2)}$ equal to their long run means. The baseline estimated parameter values for the dynamics of $X^{(5)}$ are $\theta_6 = 0.000254$, $\kappa_5 = 1.005073$, $\sigma_5 = 0.000542$, $\eta_5 = 0.000269$, $\kappa_{5,3} = 0.003648$, $\kappa_{5,4} = 0.000018$, $\kappa_{5,6} = 0.000261$.

<table>
<thead>
<tr>
<th>$\sigma_5$</th>
<th>$p_{25}$</th>
<th>$\bar{p}^{*}$</th>
<th>spread (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0005</td>
<td>0.201425</td>
<td>0.201469</td>
<td>2.15</td>
</tr>
<tr>
<td>0.0100</td>
<td>0.201425</td>
<td>0.201822</td>
<td>0.201822</td>
</tr>
<tr>
<td>0.0150</td>
<td>0.201425</td>
<td>0.202009</td>
<td>28.96</td>
</tr>
<tr>
<td>0.0200</td>
<td>0.201425</td>
<td>0.202196</td>
<td>38.26</td>
</tr>
<tr>
<td>0.1000</td>
<td>0.201425</td>
<td>0.205237</td>
<td>189.24</td>
</tr>
<tr>
<td>0.1500</td>
<td>0.201425</td>
<td>0.207184</td>
<td>285.90</td>
</tr>
</tbody>
</table>
Figure 1: Survival curves computed at the beginning of each year $t = 1980, \ldots, 2004$ for England & Wales males aged $65 + t - 1980$ in year $t$. Forecasts are based on the Lee-Carter model using the latest Human Mortality Database data available at the beginning of each year $t$. 
Figure 2: Mark-to-market value of the longevity swap (MTM) and stream of cashflows with no credit risk (CFs), and with counterparty B’s credit spreads widening by 25 and 50 basis points over 1988-2005.
Figure 3: Swap margins $p_{T_i}/p_{T_{i-1}} - 1$ computed for different maturities $\{T_i\}$ and collateral rules, with $\delta^A = \lambda^A$ and $\delta^B = \lambda^B$: no collateral (squares), full collateralization (circles); $\lambda^A = \lambda^B + \Delta$, with $\Delta = 0$ (dashed lines) and $\Delta = 0.01$ (solid lines). The underlying is a cohort of 10,000 US males aged 65 at the beginning of 2008. Swap rates are plotted against the percentiles of improvements in survival rates based on Lee-Carter forecasts.
Figure 4: Swap margins $p_{T_i}/p_{T_{i-1}} - 1$ computed for different maturities $\{T_i\}$ and collateral rules, with $\delta^A = \lambda^A = \lambda^B + 0.01$ and $\delta^B = \lambda^B$: no collateral (squares), full collateralization (circles), full collateral posted only by party A (stars) or party B (diamonds). The underlying is a cohort of 10,000 US males aged 65 at the beginning of 2008. Swap rates are plotted against the percentiles of improvements in survival rates based on Lee-Carter forecasts.
Figure 5: Swap margins $\frac{p_{T_i}}{p_{T_i}} - 1$ computed for different maturities $\{T_i\}$ and collateral rules, with $\lambda^A = \lambda^B$ and $\delta = X^{(5)}$, where the parameter estimates of $X$ are given in table 2. Collateral rules: no collateral (squares), full collateralization (circles), full collateral posted only by party $A$ (stars) or $B$ (diamonds). Swap rates are plotted against the percentiles of improvements in survival rates based on Lee-Carter forecasts.