Exit time and occupation time problems for insurance risk processes

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20th IME Congress
Atlanta, Georgia
July 24-27, 2016
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Outline

1 Introduction
   - Background - Ruin theory
   - Gerber-Shiu analysis

2 Finite-time ruin problems
   - Probabilistic approach
   - Analytic approach

3 Exotic ruin problems
   - Parisian ruin
   - Drawdown
   - Occupation times
   - Discrete Poisson observations

4 Conclusion
Introduction

Finite-time ruin problems

Exotic ruin problems

Conclusion

Background - Ruin theory

Insurance risk processes

- Analysis of stochastic processes of spectrally negative form (known as insurance risk processes)
  - downward jumps
  - no upward jumps

- Long and rich history: Pioneer work by Lundberg (1903) and Cramér (1930)

- Reflect the nature of the insurance business (e.g., claim payments, continuous stream of premium income)

- This contrasts with financial applications:
  - processes with both upward and downward jumps in the underlying asset are desirable
Standard exit problem: ruin theory
Insurance risk processes

- Notable examples of insurance risk processes \( \{U_t\}_{t \geq 0} \):
  - Cramér-Lundberg risk process
  - Spectrally negative Lévy process
  - Sparre Andersen risk process
  - Spectrally negative Markov-additive process (MAP)
The Cramér-Lundberg risk process is defined as

\[ U_t = x + ct - S_t, \]

where \( c > 0 \) and \( \{S_t\}_{t \geq 0} \) is a compound Poisson process with

- claim arrival rate \( \lambda > 0 \)
- iid claim sizes with density \( p \), df \( P(x) = 1 - \bar{P}(x) \) and LT \( \bar{p} \)
Standard exit problem: ruin theory

- First downward exit of the insurance risk process $\{U_t\}_{t \geq 0}$ below a given level $b \geq 0$
  - Exit time (i.e., Ruin time)
    $$\tau_b^- = \inf \{ t \geq 0 : U_t < b \}$$
  - Exit overshoot (i.e., Deficit at ruin)
    $$b - U_{\tau_b^-}$$
  - Risk level just prior to the exit (i.e., Surplus immediately prior to ruin)
    $$U_{\tau_b^-}$$

See Biffis & Morales (2010) and Cheung, L., Willmot and Woo (2010) for other quantities of interest related to $\tau_b^-$
Standard exit problem: ruin theory

Of particular interest for risk management purposes:

- Indication of an insurer’s ability to sustain a given surplus level
  - Policyholders
  - Stockholders
  - Regulators and governmental bodies
  - General public

- Better understand inability to meet future obligations
  - Barometers: Timing and severity of capital shortfall
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Gerber-Shiu functionals

- Without loss of generality, assume \( b = 0 \)
- Gerber-Shiu potential measure

\[
K^{(\delta)} (x, dy, dz) := \mathbb{E}_x \left[ e^{-\delta \tau^-_0} ; -U_{\tau^-_0} \in dy, U_{\tau^-_0}^- \in dz \right], \quad x, y, z \geq 0
\]

- Gerber-Shiu function

\[
m^{(\delta)} (x) := \mathbb{E}_x \left[ e^{-\delta \tau^-_0} w \left( U^-_{\tau^-_0}, -U_{\tau^-_0}^- \right) 1\{\tau^-_0 < \infty\} \right]
\]

\[
= \int_0^\infty \int_0^\infty w (z, y) K^{(\delta)} (x, dy, dz)
\]
Gerber-Shiu analysis

Gerber-Shiu functionals

- Laplace transform (LT) of the time to ruin
  \[ \phi_{\delta}(x) = \mathbb{E}_x \left[ e^{-\delta \tau^-_0} 1_{\{\tau^-_0 < \infty\}} \right] \]

- Infinite-time ruin probability
  \[ \psi(x) = \mathbb{P}_x \left( \tau^-_0 < \infty \right) \]

- Finite-time ruin probability
  \[ \psi(x, t) = \mathbb{P}_x \left( \tau^-_0 \leq t \right) \]

- Trivariate LT
  \[ \Phi_{\delta, z, s}(x) = \mathbb{E}_x \left[ e^{-\delta \tau^-_0 - z \tau^-_0 - s \tau^-_0} 1_{\{\tau^-_0 < \infty\}} \right] \]
Comprehensive treatment of Gerber-Shiu functions in the literature:

- **Spectrally negative Lévy process**: Doney & Kyprianou (2006), Biffis & Kyprianou (2010), Biffis & Morales (2010), Kyprianou (2013)
The timing of a capital shortfall is of paramount importance for risk management purposes.

Interest in the derivation of explicit expressions for the distribution of the time to ruin:
- Finite-time ruin probability
- Density of the time to ruin

Comparison of the probabilistic and analytic approaches:
- Probabilistic approach (Prahbu’s or Seal’s formula)
- Analytic approach (Lagrange expansion)
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Probabilistic approach

Prahbu’s (or Seal’s) formula

- Probabilistic arguments: Prabhu (1971), Seal (1978)
- Ruin can occur by time $t$
  - surplus level at time $t$ is negative
  - surplus level at time $t$ is positive, but the process has at least one downcrossing of level 0 prior to $t$
- condition on the time of the last upcrossing of level 0 in $[0, t]$
- utilize the skip-free upward and strong Markov properties of the Cramér-Lundberg risk process
Prahbu’s (or Seal’s) formula

- **Finite-time ruin probability**
  \[
  \psi (x, t) = \mathbb{P} (S_t > x + ct) + \int_0^t c \mathbb{P} (S_w \in x + cdw) (1 - \psi (0, t - w))
  \]

- **Ballot theorem** (e.g., Feller (1966), Takács (1967) and Gerber (1988))
  \[
  \mathbb{P} (S_w \leq cw \text{ for } \forall w \in [0, t] , \ S_t \in dy) = \frac{ct - y}{ct} \mathbb{P} (S_t \in dy)
  \]

- **Survival probability** (with \( u = 0 \))
  \[
  1 - \psi (0, t) = \int_{0^-}^{ct} \frac{ct - y}{ct} \mathbb{P} (S_t \in dy)
  \]
Probabilistic approach

Prahbu’s (or Seal’s) formula

- Prabhu’s (or Seal’s) arguments (e.g., Dickson (2007))

\[ w(x, t) := \frac{\partial}{\partial t} \psi(x, t) \]

\[ = \int_{0}^{x+ct} \mathbb{P}(S_t \in dy) \lambda \bar{P}(x + ct - y) \]

\[ - \int_{0}^{t} c \mathbb{P}(S_w \in x + cdw) w(0, t - w) \]
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Alternative approach to derive the df of the time to ruin

- Utilize a Lagrangian identity to invert the LT of the time to ruin (e.g., Dickson and Willmot (2005))
- Gain inherent structure through the incorporation of
  - Number of claims until ruin $N_{\tau^-}$
  - Aggregate claim until ruin $S_{\tau^-}$ (equivalently, the deficit at ruin)
Gerber-Shiu function

\[
\phi_{r,\delta,z}(x) \coloneqq \mathbb{E}_x \left[ r^{N_{\tau_0^-}} e^{-\delta \tau_0^- - z S_{\tau_0^-}} 1\{\tau_0^- < \infty\} \right]
= \sum_{n=1}^\infty \int_0^\infty \int_0^\infty r^n e^{-\delta t - z y} \mathbb{P}_x \left( \tau_0^- \in dt, N_{\tau_0^-} = n, S_{\tau_0^-} \in dy \right)
\]

Its Laplace transform satisfies

\[
\tilde{\phi}_{r,\delta,z}(s) = \lambda r \tilde{p}(z) \left( \tilde{P}_z(\rho) - \tilde{P}_z(s) \right) \tilde{v}_{r,\delta,z}(s)
\]

"scale function" \( \{v_{r,\delta,z}(u)\}_{u \geq 0} \) with LT

\[
\tilde{v}_{r,\delta,z}(s) = \frac{1}{cs - \lambda - \delta + \lambda r \tilde{p}(z) \tilde{p}_z(s)}
\]

\( \rho \) is the unique (non-negative) zero of \( 1/\tilde{v}_{r,\delta,z}(s) \)
Analytic Lagrangian inversion

- Initial value theorem
  \[ \phi_{r,\delta,z}(0) = \frac{\lambda r \tilde{p}(z) \tilde{P}_z(\rho)}{c} \]

- Application of a Lagrangian identity for inversion (e.g., Benes (1957), Panjer and Willmot (1992))
  \[ e^{-\rho y} = e^{-\left(\lambda + \delta\right)\frac{y}{c}} \]
  \[ + \sum_{n=1}^{\infty} r^n \int_{\frac{y}{c}}^{\infty} e^{-\delta t - z(\rho - y)} \left\{ \frac{y e^{-\lambda t} (\lambda t)^n}{t^n} p^* (ct - y) \right\} dt \]
Analytic Lagrangian inversion

- Ruin law \((x = 0)\)

\[
\mathbb{P}\left( \tau_0^- \in dt, N_{\tau_0^-} = n, S_{\tau_0^-} \in dy \right)
= \begin{cases} 
\lambda e^{-\lambda t} p(y) \, dy, & n = 1, \\
\tau_{\lambda,n}(t) \int_0^t \frac{ct-z}{ct} p^{n-1}(z) p(y-z) \, dz, & n = 2, 3, \ldots
\end{cases}
\]

for \(y > ct \geq 0\) where

\[
\tau_{\lambda,n}(t) = \lambda \frac{(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!}, \quad t \geq 0
\]
Special case: density of the time to ruin (e.g., Dickson and Willmot (2005))

\[ w(0,t) = \lambda e^{-\lambda t} \overline{P}(ct) \]
\[ + \sum_{n=2}^{\infty} \tau_{\lambda,n}(t) \int_{0}^{ct} \frac{ct-z}{ct} p^{*(n-1)}(z) \overline{P}(ct-z) \, dz \]

Similar idea for the ruin law when \( x > 0 \)
The Lagrangian methodology can be extended to handle problems of a more complex nature:
- show great versatility
- relevant in ruin problems involving multiple solutions of Lundberg’s generalized equation

Notable applications of Lagrange’s multivariate theorem:
- Time to ruin: Sparre Andersen risk model with
  - combination of exponential interclaims: Shi & L. (2013)
  - exponential jumps: L., Shi & Willmot (2011)
- Two-sided exit problem:
Analytic Lagrangian inversion

- Immediate applications in queuing theory in the study of certain first passage times (e.g., busy period)
  - Fluid flow connection between risk processes and queues
    - Asmussen (1995)
    - Ahn, Badescu & Ramaswami (2007)
  - Example: M/G/1 queue - renewal model with exponential claims
Analytic Lagrangian inversion
Exotic ruin problems

- Parisian ruin
- Drawdowns
- Occupation times
- Discrete Poisson observations
- Dividend problem
- Tax problem
Exotic ruin problems

- Focus exclusively on the Cramér-Lundberg risk process
  - simplicity of the idea
  - ease of presentation

- The extension to the more general spectrally negative Lévy process (loosely speaking, from the bounded to unbounded variation case) can be done using one of the following techniques:
Spectrally negative Lévy process

- **Approach #1**: A spatial approximation methodology
  - spatial shift of the underlying process (or equivalently certain stopping times) by \( \pm \varepsilon \)
  - construct bounds to the quantity under study
  - show convergence when \( \varepsilon \downarrow 0 \)

Spectrally negative Lévy process (cont’d)

- **Approach #2**: Convergence of processes approximation
  - introduce a sequence of bounded variation processes $\{X_n\}_{n \geq 1}$ which converge to the unbounded variation process $X$ (e.g., Bertoin (1996))
  - derive the result in the bounded variation case
  - obtain convergence results when $n \to \infty$

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Parisian ruin

- Relax the definition of the ruin event
- Allow a grace period for the insurer’s surplus process to recover to an acceptable capital level
- Of practical importance under Chapters 7 and 11 of the US bankruptcy code (and similarly in other jurisdictions) (see Li, Tang, Wang & Zhou (2014))
Parisian ruin

- **Grace period:**
  - **Deterministic:** a fixed period of length $\beta > 0$ (e.g., Dassios and Wu (2009), Loeffen, Czarna & Palmowski (2013))
  - **Stochastic:** randomize the grace period through an exponential time horizon (e.g., L., Renaud & Zhou (2014), Baurdoux et al. (2016))

- Define the **Parisian ruin time** as
  \[
  \xi_\beta = \inf \left\{ t > \beta : t - g_t > \beta \right\},
  \]
  where
  \[
  g_t = \sup \left\{ 0 \leq s \leq t : U_s \geq 0 \right\}
  \]

- Analyze the ruin quantity
  \[
  \nu_{\delta, z}(x) := \mathbb{E}_x \left[ e^{-\delta \xi_\beta} e^{-z \beta} | U_{\xi_\beta} | 1\{\xi_\beta < \infty\} \right]
  \]
Parisian ruin

\[ U_t \]

\[ x \]

\[ \tau_0 \quad \beta \quad \zeta_\beta \]

\[ t \]
Parisian ruin

- Key ingredient to the analysis of the Parisian ruin time $\xi_b$:
  Kendall’s identity

- Kendall’s identity provides the distribution of the first upward crossing of a level $y$

$$\tau_y^+ = \inf \{ t > 0 : U_t > y \},$$

for a spectrally negative Lévy process

**Theorem**

*For a spectrally negative Lévy process $\{X_t\}_{t \geq 0}$,*

$$\mathbb{P} \left( \tau_y^+ \in dt \right) dy = \frac{y}{t} \mathbb{P} \left( X_t \in dy \right) dt$$
Parisian ruin

- Condition on the first excursion below 0, namely

\[ \mathbb{E}_x \left[ e^{-\delta \tau_0^-} ; \left| U_{\tau^-} \right| \in dy \right] \]

(see Gerber & Shiu (1998))

- From level \(-y\), sample paths are categorized into:
  - Recovery to level 0 before \(\beta\)
  - No recovery to level 0 before \(\beta\) (i.e., length of excursion exceeds \(\beta\))

- Analysis:

\[
v_{\delta,z}(x) = \int_0^\infty \mathbb{E}_x \left[ e^{-\delta \tau_0^-} ; \left| U_{\tau^-} \right| \in dy \right] \left\{ \int_0^\beta e^{-\delta t} \mathbb{P} \left( \tau_+^y \in dt \right) v_{\delta,z}(0) + e^{-\delta \beta} \mathbb{E} \left[ e^{-z(y-U_\beta)} ; \tau_+^y > \beta \right] \right\}
\]
Parisian ruin

- For $x = 0$,

$$v_{\delta, z}(0) = \frac{\lambda e^{-\delta \beta} \int_0^\infty T_{\rho p} (y) \mathbb{E} \left[ e^{-z(y-U_\beta)} ; \tau_+^y > \beta \right] dy}{1 - \frac{\lambda}{c} \int_0^\infty T_{\rho p} (y) \int_0^\beta e^{-\delta t} \mathbb{P} \left( \tau_+^y \in dt \right) dy}$$

- Kendall’s identity is used to analyze the first excursion below 0
  - Restatement:
    $$\mathbb{P} \left( \tau_+^y \in dt \right) dy = \frac{y}{t} \mathbb{P} (S_t \in ct - dy) dt$$

- Application:
  $$\mathbb{E} \left[ e^{-z(y-U_\beta)} ; \tau_+^y > \beta \right] = \mathbb{E} \left[ e^{zU_\beta} \right] \left( e^{-zy} - \mathbb{P} \left( \tau_+^y \leq \beta \right) \right)$$

- See Loeffen, Czarna & Palmowski (2013) for the analysis in the spectrally negative Lévy process
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Introduction to Drawdown

- **Time to ruin**
  - First passage time below a given capital requirement
  - Static capital requirement/objective over time
  - Limited path-dependent information on how ruin occurs

- **Drawdown Time**
  - Triggers when a significant drop in surplus level (wrt historical highs) occurs
  - Incorporates more path-dependent features of the process
  - For large initial surplus, early warning mechanism to a potential ruin event
Introduction to Drawdown

- **Drawdown** measures the *decline* in value from historical peak.

- For a process \(\{U_t\}_{t \geq 0}\), we define its maximum-reflected process \(\{Y_t\}_{t \geq 0}\) (also, known as the **drawdown process**) as:

\[
Y_t := M_t - U_t,
\]

where

\[
M_t = \sup_{0 \leq s \leq t} U_s
\]
Introduction to Drawdown

- The first time the drawdown level exceeds a pre-specified level \( a > 0 \) is denoted by

\[
T_a^+ := \inf\{t > 0 : Y_t \geq a\}
\]
Applications: wide range

- Mutual fund
  - Drawdown is a frequently quoted investment performance measure (Schuhmacher & Eling, 2011)
  - Drawdown is an alternative measurement for volatility

- Financial mathematics
  - Option pricing: Russian option where the optimal exercise time is a drawdown time
Applications: wide range

- **Statistics**

- **Probability**
Drawdown

Drawdown analysis

- Quantities related to the drawdown time $T_a^+$:
  - $Y_{T_a^+}$: drawdown level at $T_a^+$
  - $M_{T_a^+}$: current running maximum at $T_a^+$
  - $G_{T_a^+}$: last time the process $\{U_t\}_{t \geq 0}$ was at the running maximum before $T_a^+$

- $G_{T_a^+}$ is the turning point from rising to crashing
Drawdown analysis

- Analysis of the quadruple LT

\[ \gamma_m(y) := \mathbb{E}_{y,m} \left[ e^{-\delta T^+_a - s(Y_{T^+_a} - a) - zM_{T^+_a} - qG_{T^+_a}} \right], \]

where \( \mathbb{E}_{y,m} \) is the expectation under \( Y_0 = y \) and \( M_0 = m \)

- Consider the Cramér-Lundberg risk process \( \{U_t\}_{t \geq 0} \)
  - Drawdown process \( \{Y_t\}_{t \geq 0} \) is a M/G/1 queue with initial workload \( y \geq 0 \)
  - Primary quantity for the drawdown analysis: two-sided exit result (e.g., \textbf{L.}, Li & Zhang (2016))
Drawdown analysis

- No less general to consider $Y_0 = 0$

\[ \gamma_m(0) := \mathbb{E} \left[ e^{-\delta T^+_a - s(Y^+_a - a) - z(m + c\kappa^+_a) - qG^+_a} \right] \]

where

\[ \kappa^+_a = \int_0^{T^+_a} 1\{Y_t = 0\} dt \]

- Condition on the first jump and subsequently the first exit of $Y$ from $(0,a)$:

\[ \gamma_m(0) = \lambda e^{-zm} \left( cz + \delta + q + \lambda \left( 1 - \int_0^a \frac{W^{(\delta+q)}(a-y)P(dy)}{W^{(\delta+q)}(a)} \right) \right) \]
Drawdown analysis

- \((G_{T^+_a}, M_{T^+_a})\) is independent of \((T^+_a - G_{T^+_a}, Y_{T^+_a})\) under the Cramér-Lundberg setup

  \(\Rightarrow\) Rising part and crashing part of drawdowns are independent in both time and level scale!!!

- This conclusion holds for more general models
Drawdown analysis

- Approaches for more general processes (spectrally negative Lévy models and time-homogeneous diffusion models)
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Occupation times

- Of risk management interest
  - provide a measure of the time spent in e.g., relatively low surplus levels
  - may affect an insurer’s ability to access new capital sources

- Some notable papers:
  - Spectrally negative Lévy process: Loeffen, Renaud & Zhou (2014)
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Discrete Poisson observations

- Bridge between periodic and continuous observations
- Yield certain occupation times in the continuous case
- Limiting case (when the Poisson observation rate $\lambda \to \infty$): recover results in the continuous case
- Some notable papers:
Conclusion

- Extensive literature on the analysis of insurance risk processes: latest trends
  - additional of features to make the insurance risk process more realistic (e.g., tax, dividends)
  - analysis of more "advanced" risk management quantities of interest

- Parallel many problems of interest in finance but for processes of spectrally negative form

- State-dependent risk processes (Omega-type risk processes): a promising direction for future research (e.g., Albrecher, Gerber & Shiu (2011), Li & Palmowski (2016))